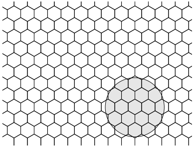
Discrete Geodesics and Cellular Automata

Gilles Dowek

Joint work with Pablo Arrighi

Cellular automata



- Each cell has a finite state space
- State of a cell depends on state of a finite number of cells at previous time step
- Local evolution function the same everywhere and everywhen

Gandy's hypotheses

- Same everywhere everywhen: homogeneity of time and space
- Finite number of cells: bounded velocity of information
- Finite state space: bounded density of information

Bounded density of information

1981: 380 km/h, 1990: 515 km/h, 2007: 574 km/h

But this has to stop (c)

2000: 1 Gb, 2009: Gb, 2013: 1 Tb

But this has to stop

Gandy - Bekenstein: a bound (h)

This is not a real number

Distances, coordinates, ... in $\mathbb{R}$ $\Delta\mathbb{Z}$: discrete spacetime

If Gandy's hypotheses are verified

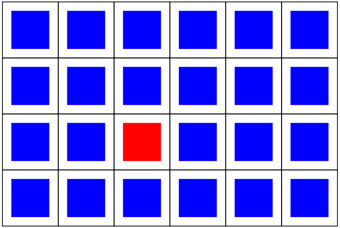
Any system (can be simulated by | is) a cellular automaton

Digital physics: model phenomena with ~~differential equations~~ cellular automata

This talk: towards gravitation (general relativity) as a cellular automaton

Motion in a cellular automaton

States: $\{q, ..., \}$
All cells are quiescent (q) except one: the particle



Evolution rules preserve this invariant

Metric

In special relativity, when changing reference frame
$x^2 + y^2$ not preserved, t^2 not preserved
But "distance" $t^2 - x^2 - y^2$ preserved

$$\begin{pmatrix} t & x & y \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} t \\ x \\ y \end{pmatrix}$$

Metric tensor

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Same everywhere
Free particle: constant velocity: spacetime straight line
A geodesic of this metric (same everywhere)

Gravitational attraction of a point mass (star)

Not a force, but a modification of the metric tensor at $\langle t, x, y \rangle$

$$\begin{pmatrix} 1 - \frac{2m}{r} & 0 & 0 \\ 0 & -\frac{x^2}{r(r-2m)} - \frac{y^2}{r^2} & -\frac{2mxy}{r^2(r-2m)} \\ 0 & -\frac{2mxy}{r^2(r-2m)} & -\frac{x^2}{r^2} - \frac{y^2}{r(r-2m)} \end{pmatrix}$$

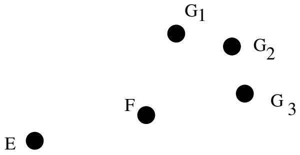
where $r = \sqrt{x^2 + y^2}$, m mass of the star in m, i.e. $\frac{\mathcal{G}}{c^2} M$

Schwarzschild's metric: solution of Einstein's eq. for point mass

What is a geodesic in a discrete spacetime?

Geodesic: shortest and straightest path between two points

$w(E, F, G)$ measures how E, F, G deviates from going straight ahead, e.g. (external) angle in F



G chosen to minimize (external) angle in F
Discrete-time continuous-space spacetime:
$E_0, E_1, E_2, E_3, \dots$ geodesic if for all i, $w(E_{i-1}, E_i, E_{i+1}) = 0$

From the continuous to the discrete

Too strong in discrete space

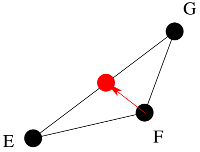
$w(E_{i-1}, E_i, E_{i+1})$ minimum for local variations of E_{i+1}: for any spatial neighbor G of E_{i+1}

$$w(E_{i-1}, E_i, G) \geq w(E_{i-1}, E_i, E_{i+1})$$

Algorithm:
E_{i-1}, E_i given
Pick random E_{i+1}
If $w(E_{i-1}, E_i, E_{i+1})$ not local minimum replace E_{i+1} by a better neighbor and iterate

Metrics and deviations

$$l(E, F, G) = d(E, F) + d(F, G)$$



In the continuous case

$$w(E, \langle t, x, y\rangle, G) =$$
$$(\partial_t l(E, \langle t, x, y\rangle, G))^2 + (\partial_x l(E, \langle t, x, y\rangle, G))^2 + (\partial_y l(E, \langle t, x, y\rangle, G))^2$$

In the discrete case: replace derivatives with finite differences

Towards a cellular automaton

Schwarzschild's metric $\longrightarrow$ distance $\longrightarrow$ deviation function $\longrightarrow$ algorithm to compute geodesics

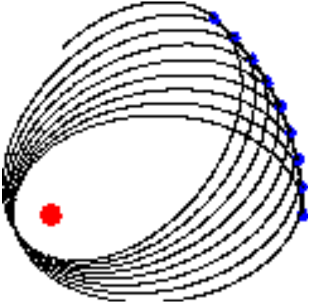
State: presence or absence of planet + value of metric tensor at this point

Locality: the velocity of the planet is bounded by c

Still big rules

But discrete

Experimental results



Almost only integers in the program
A fake planet very close to the Sun (maximize relativistic effects)
Perihelion shift 6.27°, expected 6.17°
No (t yet) similar results for Mercury: very small shift

Theoretical results

Continuous trajectories when step $\longrightarrow 0$

Unlike other discrete formulations of General relativity

Trap difficult to avoid: $(fg)' = f'g + fg'$ second term forgotten when g is taken locally constant

Conclusion

Physicists and computer scientists agree: no way to encode an unbounded amount of information in a bounded physical system

But the continuous formulations of physics implicitly assume the contrary

Physics can be reformulated in a discrete (and computational) way

Still a lot to be done:
Mercury?
More natural automata e.g. how does the planet know the star is there: messenger particles should be included