Cellular Automata: Tutorial

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Cellular Automata: examples

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The most widely known example is the **Game-of-Life** by John Conway. It is a two-dimensional CA which means that the space is an infinite checker-board. Each cell (=square of the checker-board) has two possible states, "**Alive**" and "**Dead**", represented by drawing the corresponding square black or white, respectively.

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All cells apply this rule simultaneously. As the process is repeated over and over again, a dynamical system is obtained that exhibits surprisingly complex behavior.

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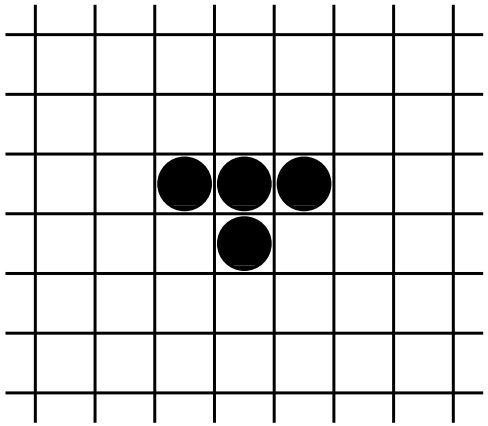

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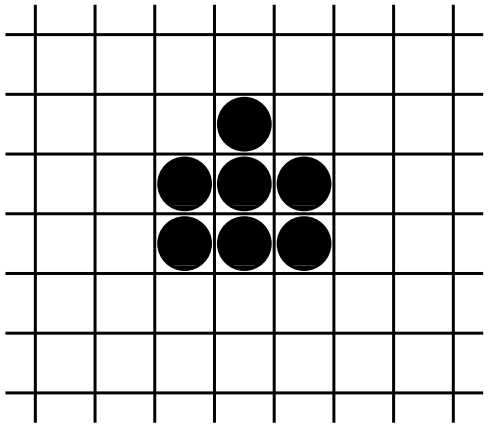

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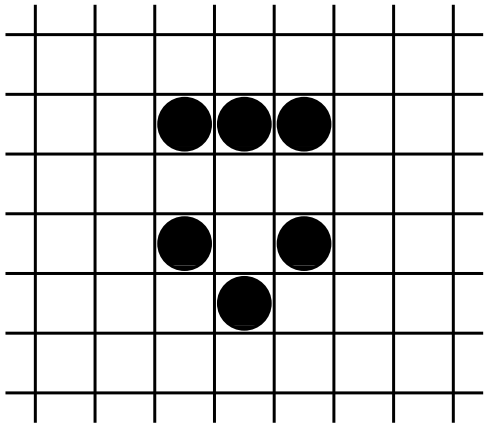

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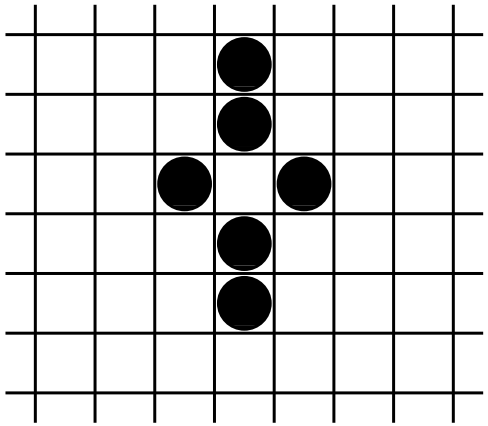

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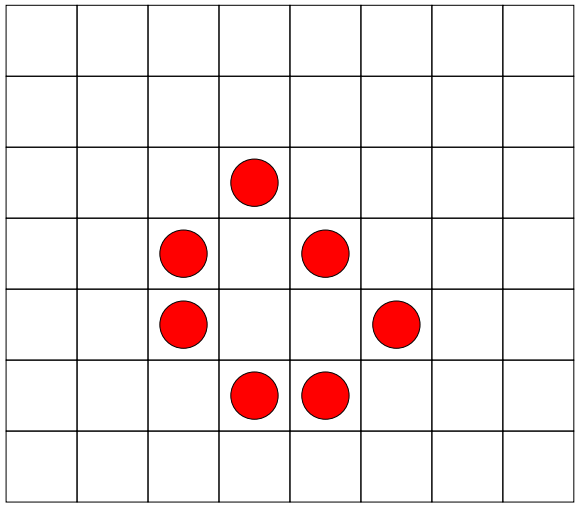

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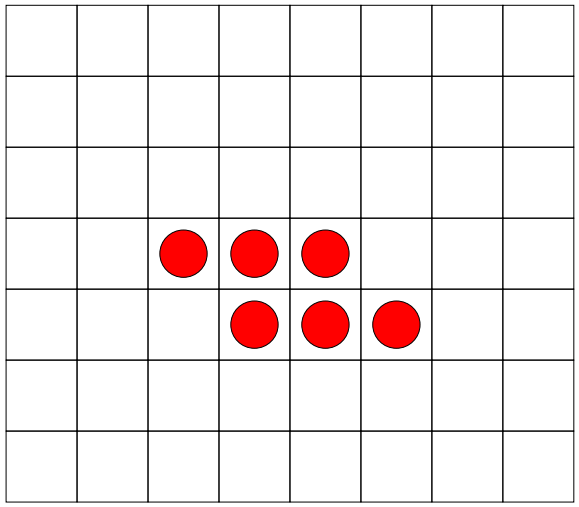

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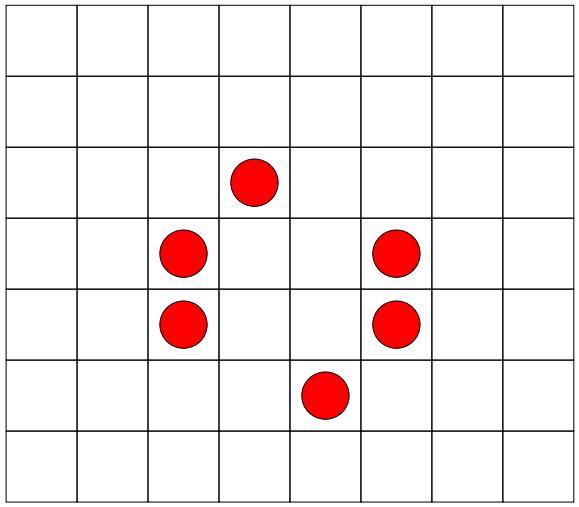

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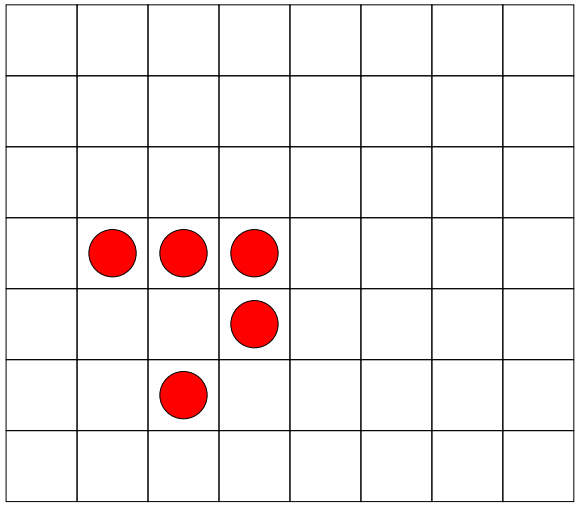

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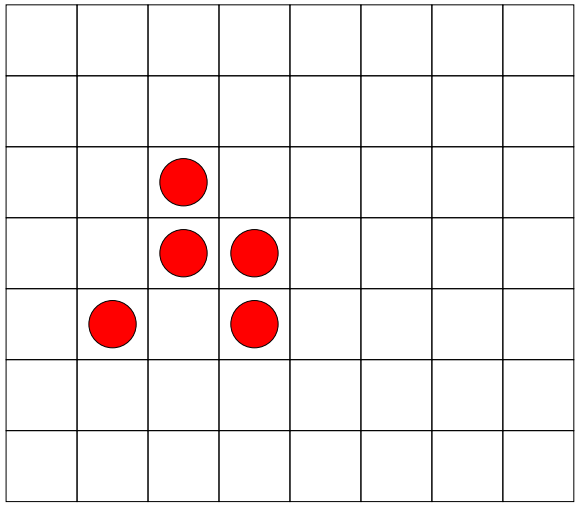

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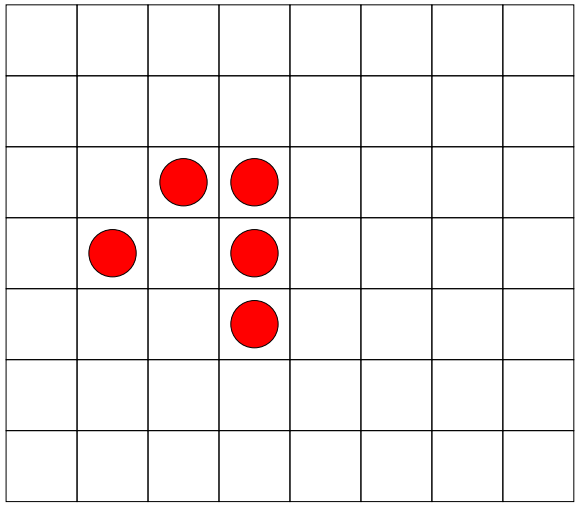

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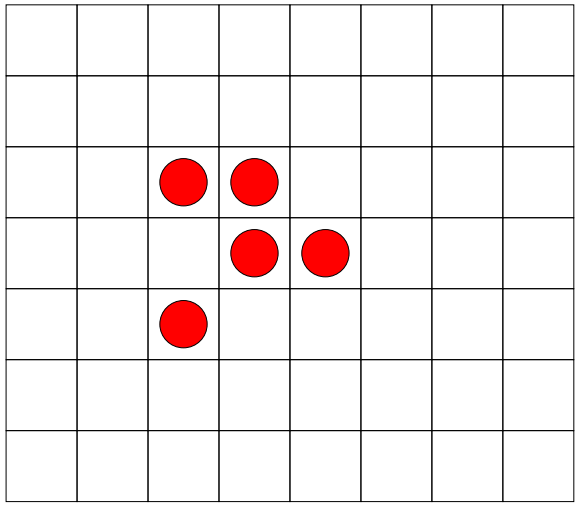

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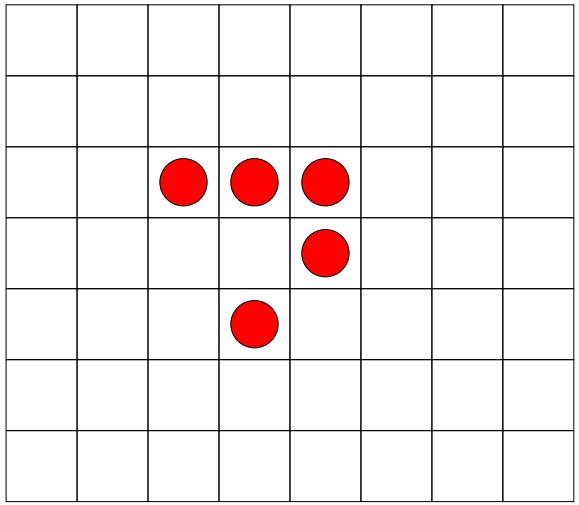

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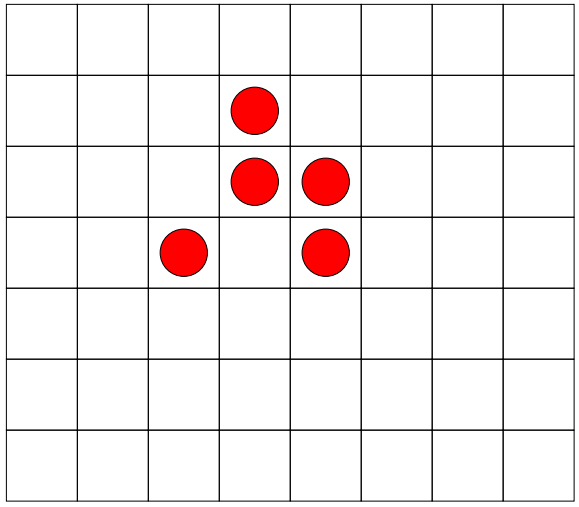

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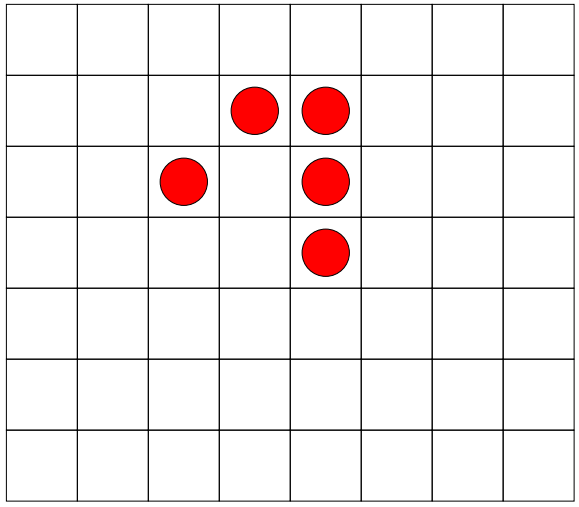

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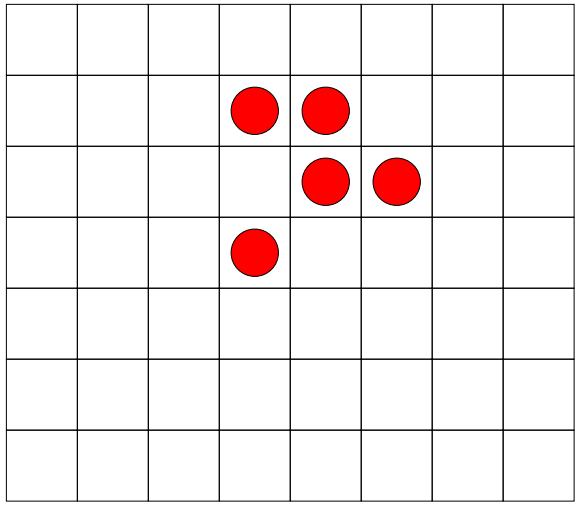

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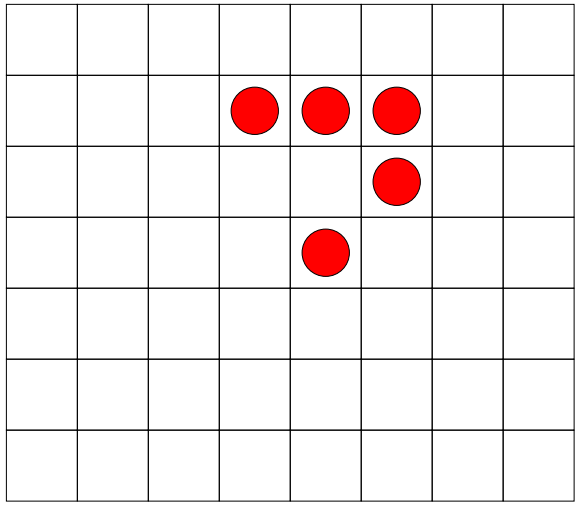

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Using such behaviors it is possible to implement information carrying signals and logical operations on these signals. It was soon discovered that Game-of-Life is a **universal** computer: for any given Turing machine M and input word w one can effectively construct a start configuration c to the Game-of-Life such that all cells eventually die if and only if M accepts w.

Another famous computationally universal CA is **rule 110** by S.Wolfram. This is a one-dimensional CA with binary state set $\{0, 1\}$, i.e. a two-way infinite sequence of 0's and 1's. Each cell is updated based on its old state and the states of its left and right neighbors as follows:

$$
\begin{array}{ccc}
111 & \longrightarrow & 0 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 1 \\
010 & \longrightarrow & 1 \\
001 & \longrightarrow & 1 \\
000 & \longrightarrow & 0
\end{array}
$$

Rule 110 was proved universal by M.Cook while working for Wolfram. Analogously to Game-of-Life, it is possible to identify signals that transmit information, and collisions that represent logic operations, but building a computer in the one-dimensional space involves solving difficult issues such as designing ways for the signals to cross each other.

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One-dimensional, binary state CA that use the nearest neighbors to determine their next state are called **elementary cellular automata**. There are only $2^8 = 256$ elementary CA, and it is quite remarkable that one of them is computationally universal.

Elementary CA were experimentally investigated in the 1980's by S.Wolfram. He designed a simple naming scheme that is still used today. The local update rule gets fully specified when one gives the next state for all 8 different contexts:

$$\begin{array}{lcl} 111 & \longrightarrow & b_7 \\ 110 & \longrightarrow & b_6 \\ 101 & \longrightarrow & b_5 \\ 100 & \longrightarrow & b_4 \\ 011 & \longrightarrow & b_3 \\ 010 & \longrightarrow & b_2 \\ 001 & \longrightarrow & b_1 \\ 000 & \longrightarrow & b_0 \end{array}$$

The **Wolfram number** of this CA is the integer whose binary expansion is

$$b_7 b_6 b_5 b_4 b_3 b_2 b_1 b_0 .$$

Elementary CA are known by their Wolfram numbers.

For example, elementary CA number 102 has local update rule

$$\begin{array}{lcl} 111 & \longrightarrow & 0 \\ 110 & \longrightarrow & 1 \\ 101 & \longrightarrow & 1 \\ 100 & \longrightarrow & 0 \\ 011 & \longrightarrow & 0 \\ 010 & \longrightarrow & 1 \\ 001 & \longrightarrow & 1 \\ 000 & \longrightarrow & 0 \end{array}$$

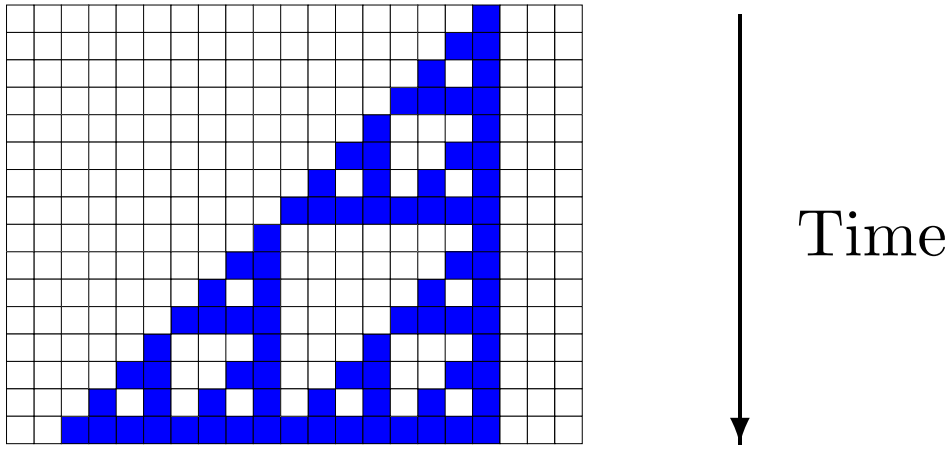
In this rule

$$a\ b\ c \longrightarrow b + c \pmod 2$$

and it will be referred to as the **xor-CA**.

A **space-time diagram** is a pictorial representation of the time evolution of the CA where consecutive configurations are drawn under each other. Horizontal lines represent space, and time increases downwards.

For example, the space-time diagram of the xor-CA starting from a single cell in state 1 is the self-similar Sierpinski-triangle:

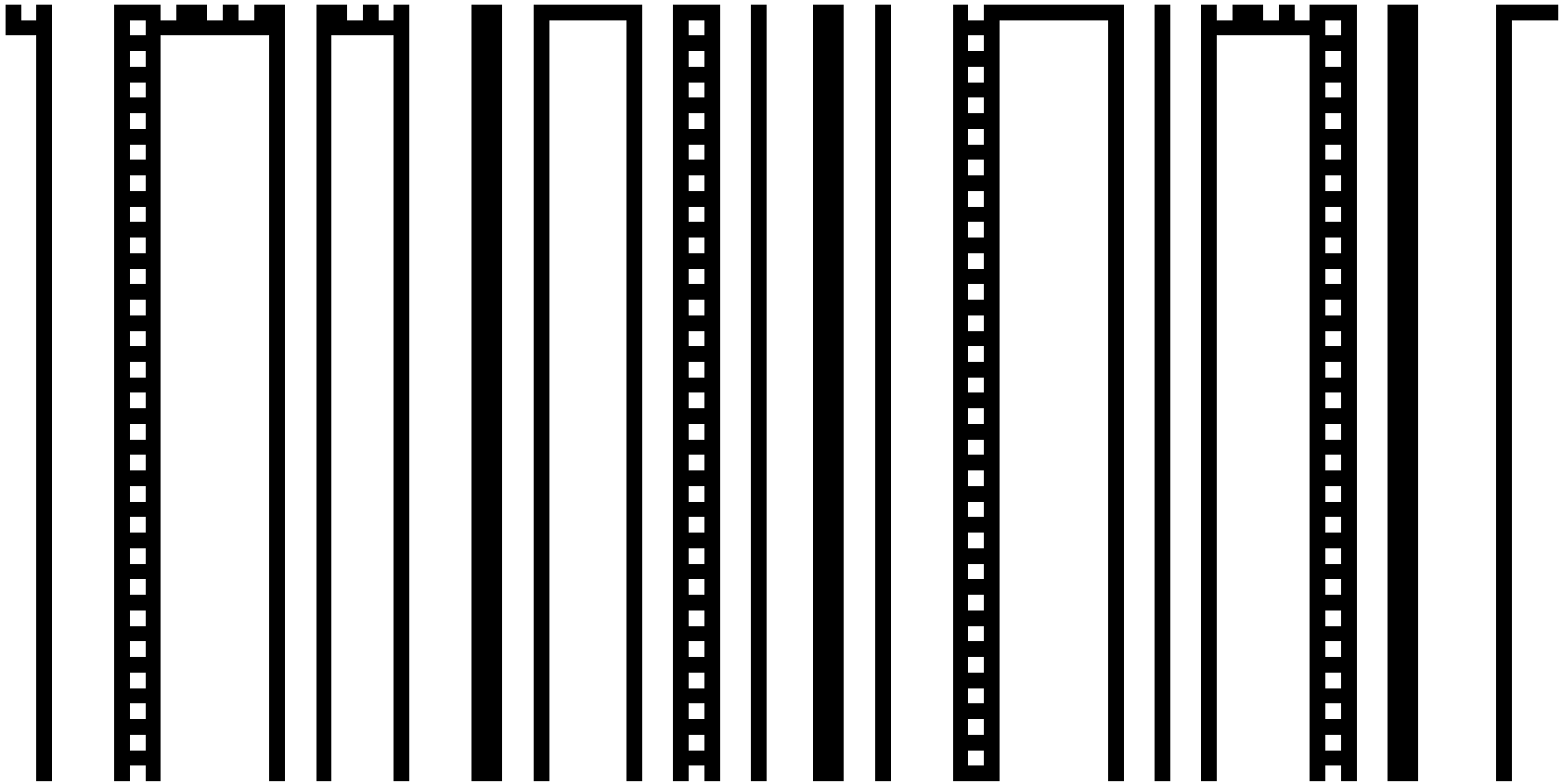


Wolfram classified elementary CA into four classes based on experiments on random initial configurations.

(W1) Almost all initial configurations lead to the same uniform fixed point configuration.

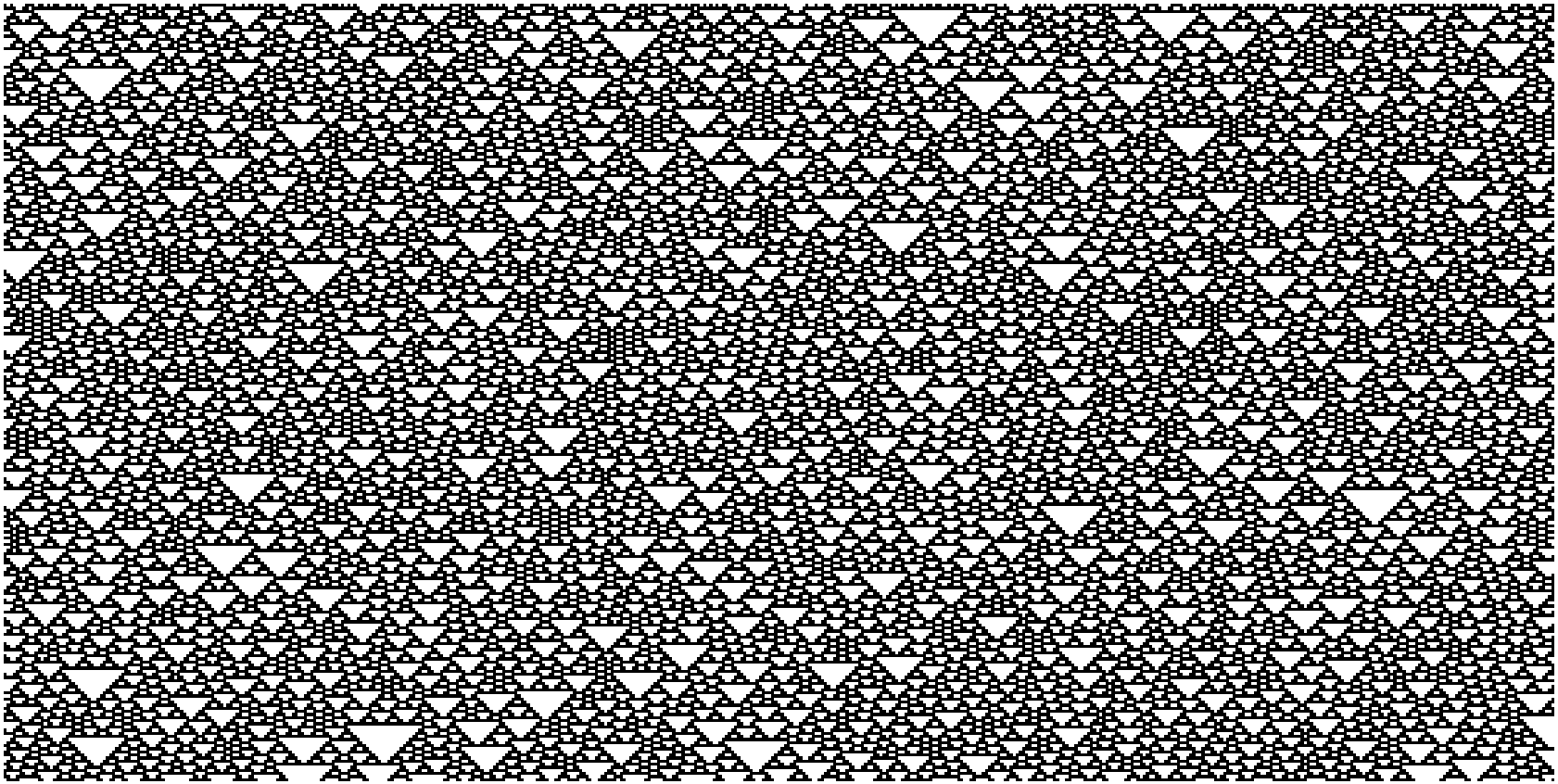
Rule 160

(W2) Almost all initial configurations lead to a periodically repeating configuration.



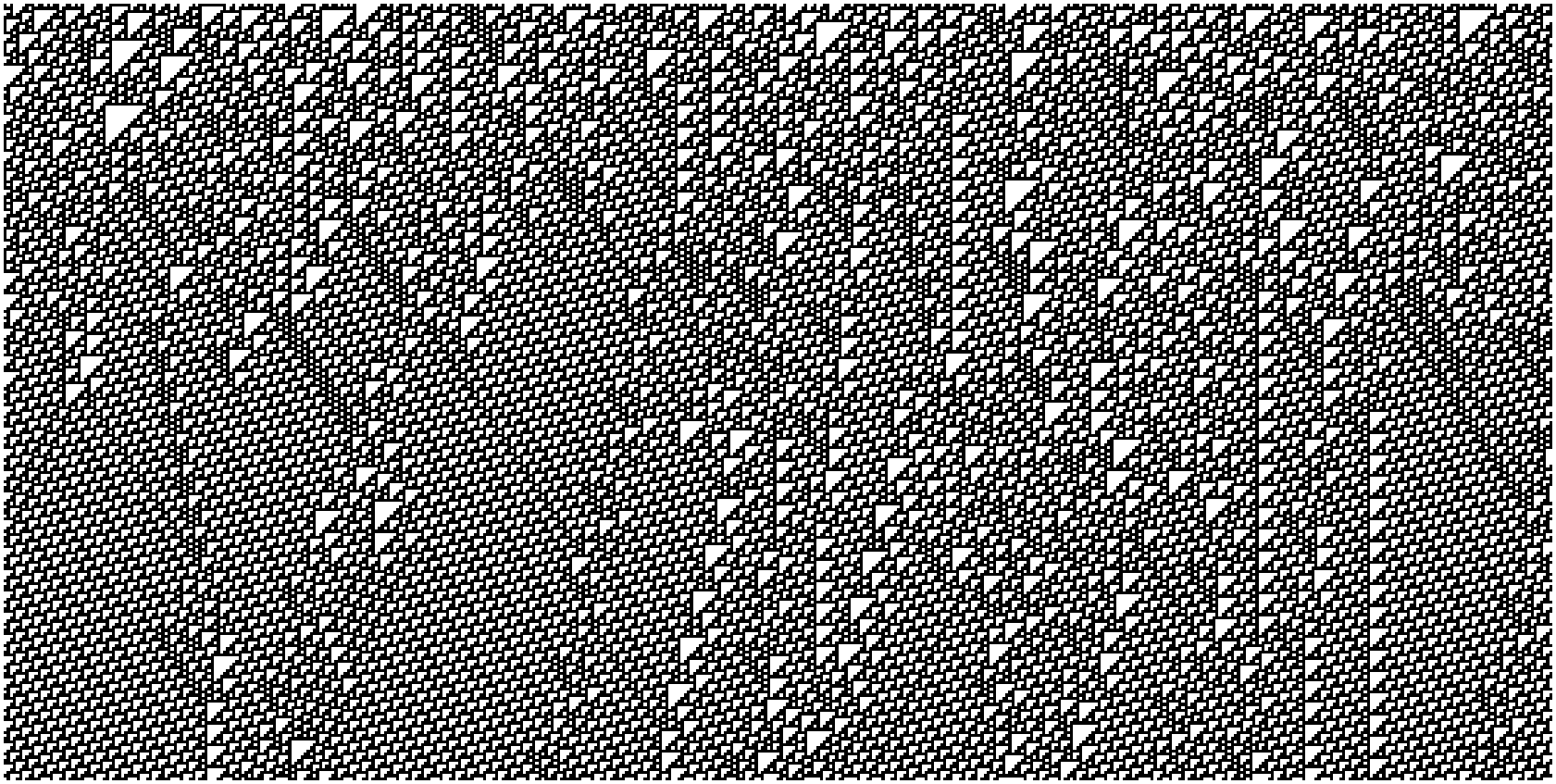
Rule 108

(W3) Almost all initial configurations lead to chaotic, "random looking" behavior.



Rule 126

(W4) Localized structures with complex interactions emerge.



Rule 110

Class (W4) has the most interesting behavior. In addition to rule 110, also rule 54 is in class (W4). It is conjectured – but not proved yet – that also rule 54 is computationally universal.

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(5) **Dynamical systems aspects** (equicontinuity classification, limit sets)

Cellular Automata: definitions

A **Cellular Automaton** (CA) is an infinite lattice of finite state machines, called **cells**. The cells of a d-dimensional CA are positioned at the integer lattice points of the d-dimensional Euclidean space, and they are addressed by the elements of $\mathbb{Z}^d$.

Let S be the finite state set. A **configuration** of the CA is a function

$$c : \mathbb{Z}^d \longrightarrow S$$

where $c(\vec{x})$ is the current state of the cell $\vec{x}$. The set $S^{\mathbb{Z}^d}$ of all configurations is uncountably infinite.

The cells change their states synchronously at discrete time steps. The next state of each cell depends on the current states of the neighboring cells according to an update rule. All cells use the same rule, and the rule is applied to all cells at the same time.

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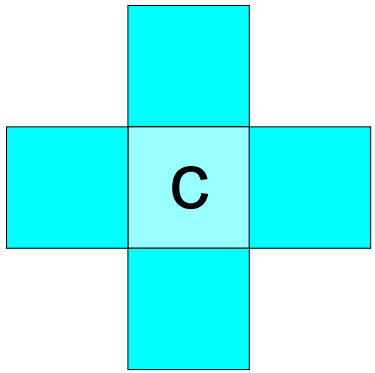
The neighboring cells may be the nearest cells surrounding each cell, but more general neighborhoods can be specified by giving the relative offsets of the neighbors. Let

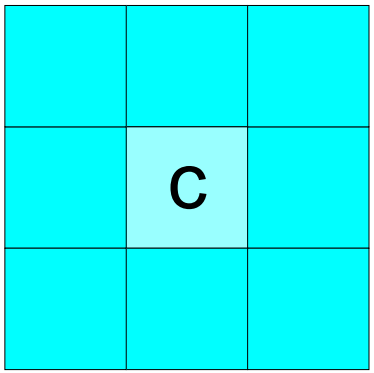
$$N = (\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n)$$

be a vector of n distinct elements of $\mathbb{Z}^d$. Then the **neighbors** of a cell at location $\vec{x} \in \mathbb{Z}^d$ are the n cells at locations

$$\vec{x} + \vec{x}_i, \text{ for } i = 1, 2, \ldots, n.$$

Typical two-dimensional neighborhoods:





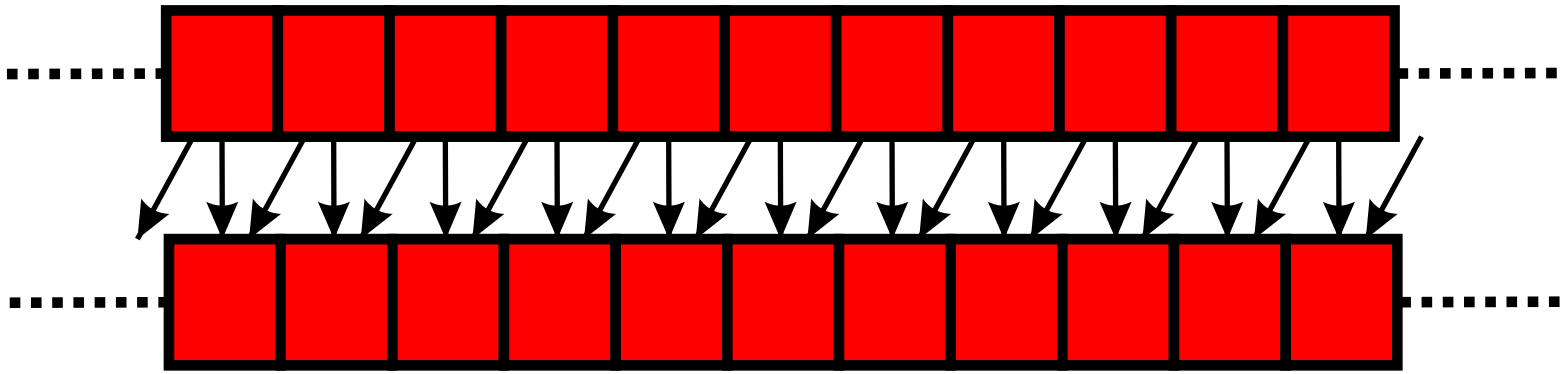
Von Neumann neighborhood

$\{(0,0),(\pm 1,0),(0,\pm 1)\}$

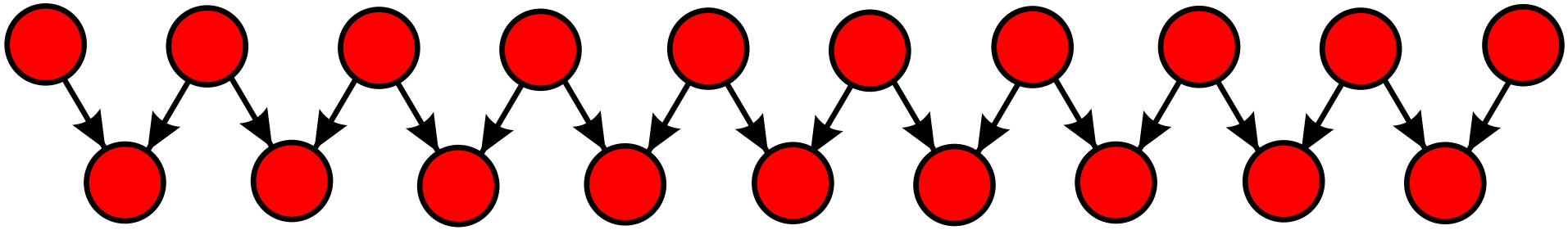
Moore neighborhood

$\{-1,0,1\} \times \{-1,0,1\}$

The smallest non-trivial one-dimensional neighborhood is the size two neighborhood $(0, 1)$. This is the neighborhood of the xor-CA. This case is **one-way** as information can not flow to the right.



If the configurations are shifted by half a cell at each step we obtain the **radius-$\frac{1}{2}$** neighborhood. The space-time diagram is then symmetric:



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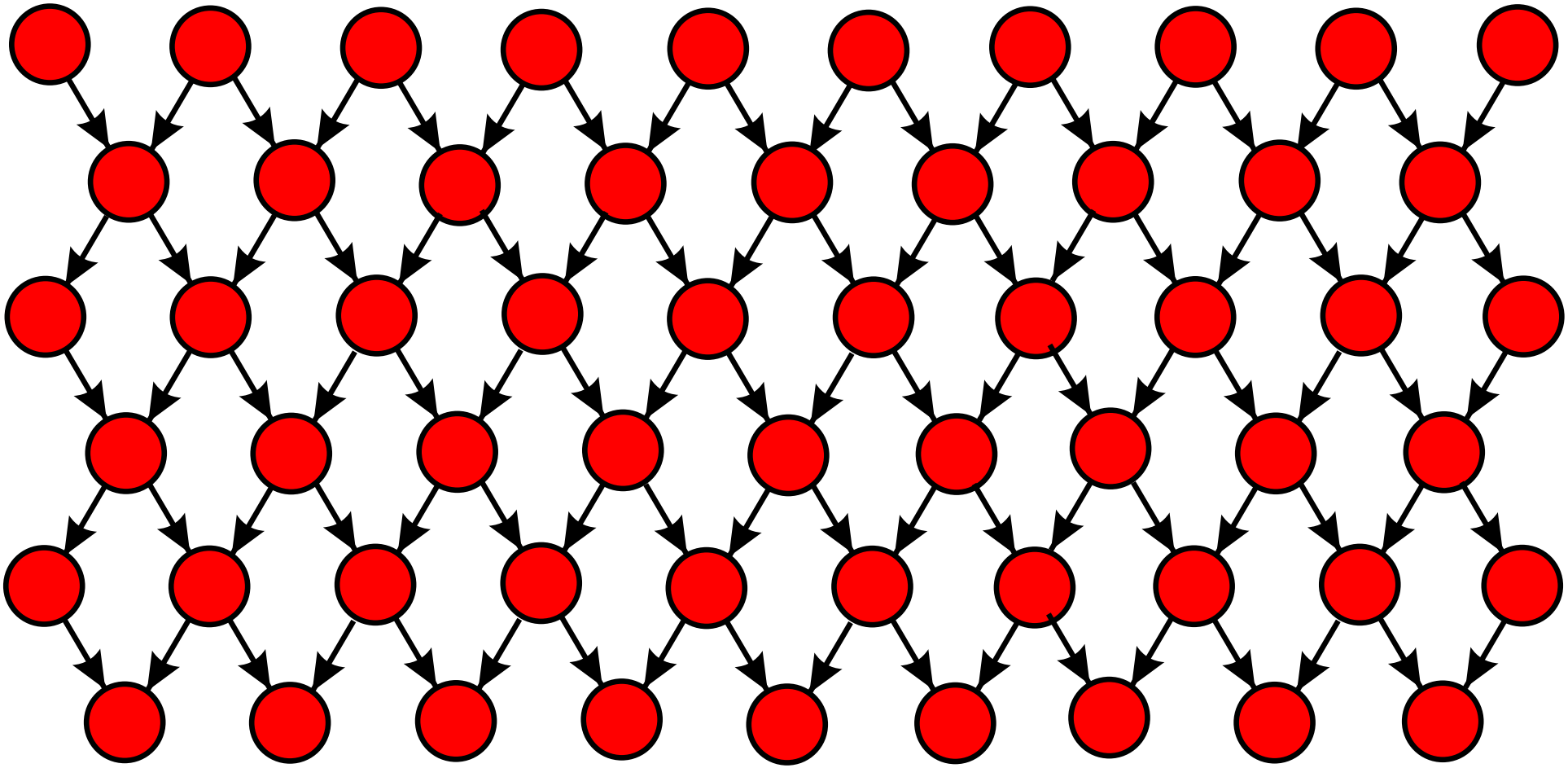

The **local rule** is a function

$$f : S^n \longrightarrow S$$

where n is the size of the neighborhood. State $f(a_1, a_2, \ldots, a_n)$ is the new state of a cell whose n neighbors were at states $a_1, a_2, \ldots, a_n$ one time step before.

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This update rule then determines the global dynamics of the CA: Configuration c becomes in one time step the configuration e where, for all $\vec{x} \in \mathbb{Z}^d$,

$$e(\vec{x}) = f(c(\vec{x} + \vec{x}_1), c(\vec{x} + \vec{x}_2), \ldots, c(\vec{x} + \vec{x}_n)).$$

We say that $e = G(c)$, and call

$$G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$$

the **global transition function** of the CA.

We typically identify the CA with its global transition function G and talk about cellular automaton function G, or simply cellular automaton G. In algorithmic questions, however, the CA is always specified by the finite items d (the dimension of the space), S (the state set), N (the neighborhood) and f (the local rule).

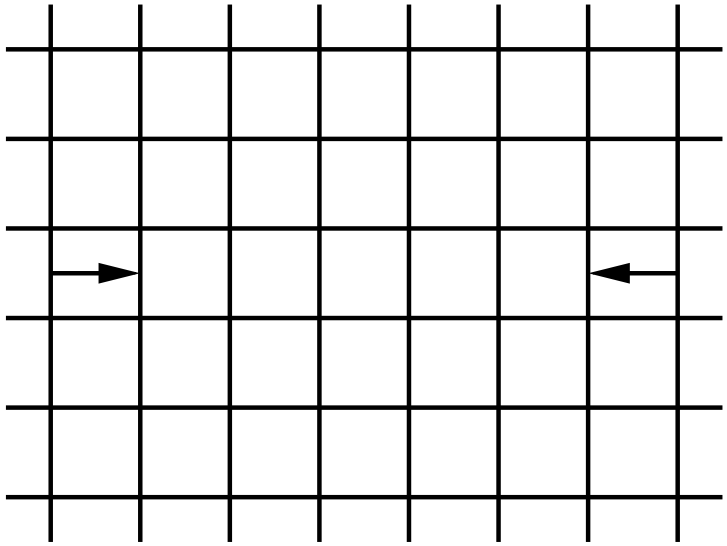
Cellular Automata can be viewed as one of the first models of natural computing. They have many properties of the physical world: They

- consist of large numbers of simple objects,
- operate in parallel,
- evolve based on local interactions,
- are homogeneous in time and space.

CA have traditionally been used in simulations of physical systems. Good examples are **lattice gases**. These are CA simulations of fluid or gas dynamics based on storing individual molecules in the cells and implementing particle interactions by the CA local rule.

Consider, for example, the simplest lattice gas model, called **HPP** (due to Hardy, Pomeau and de Pazzis). It is a two-dimensional CA where each cell can store up to four moving particles. Each particle has a direction of movement which can be up, down, left or right. There can be no more than one particle of each direction in any individual cell. So there are $2^4 = 16$ possible states.

At each time step each particle moves to the neighboring cell as indicated by the direction of the particle. If a cell receives exactly two particles moving in opposite directions then the directions of the particles are changed into the two orthogonal directions.



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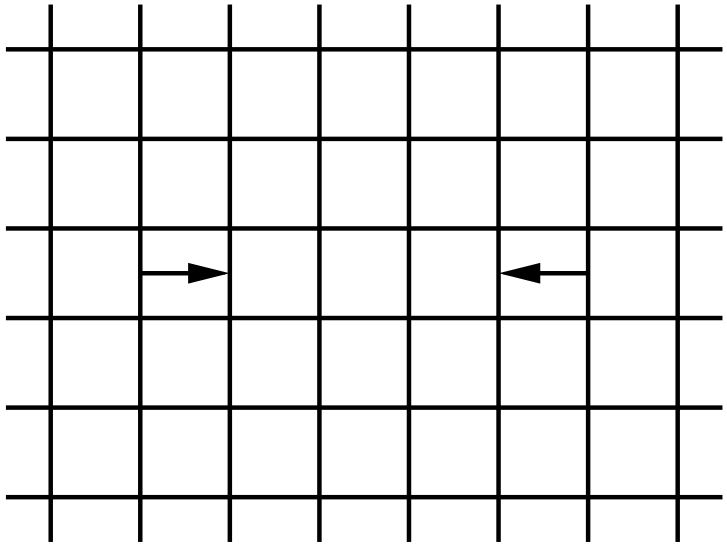

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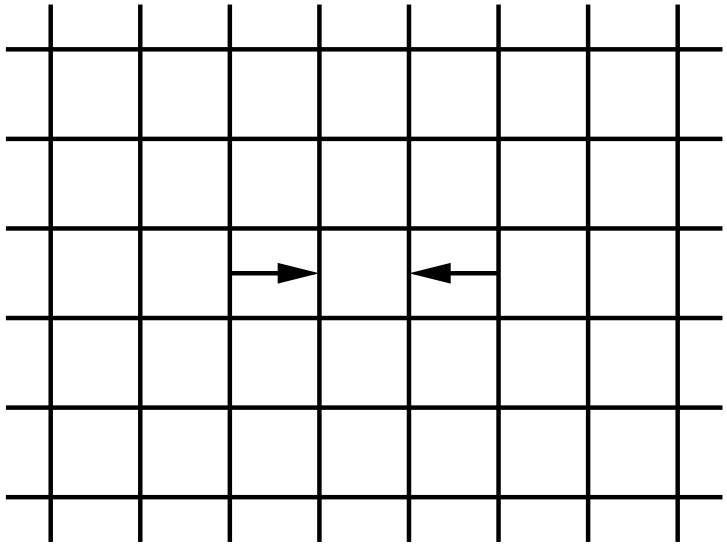

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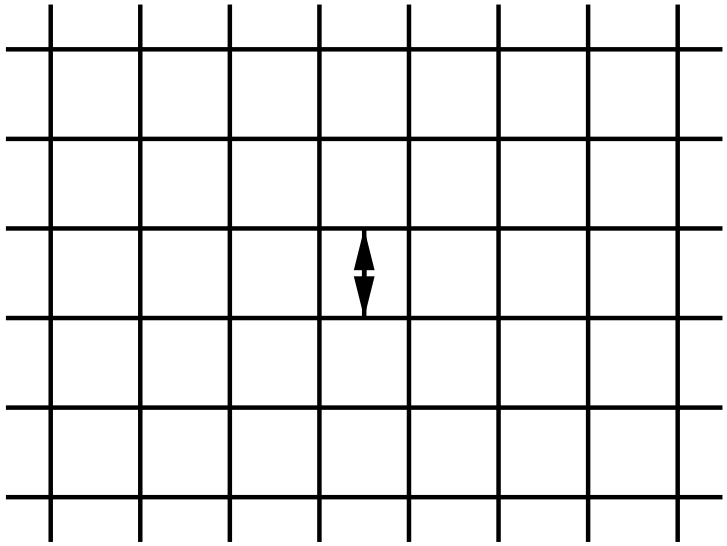

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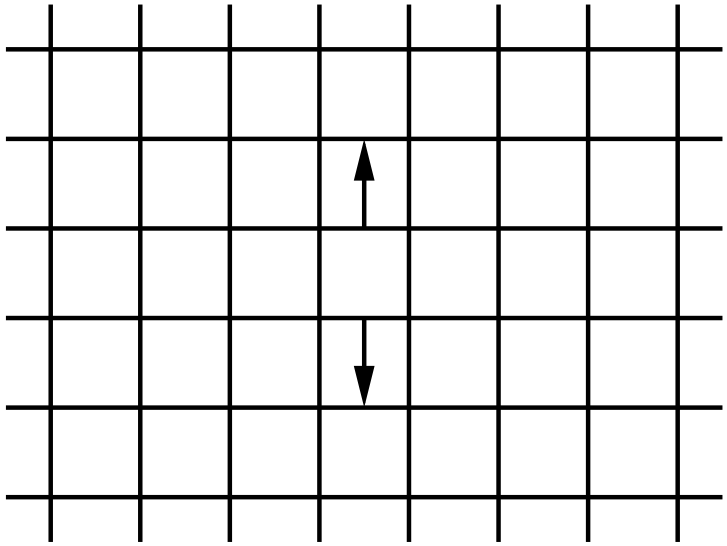

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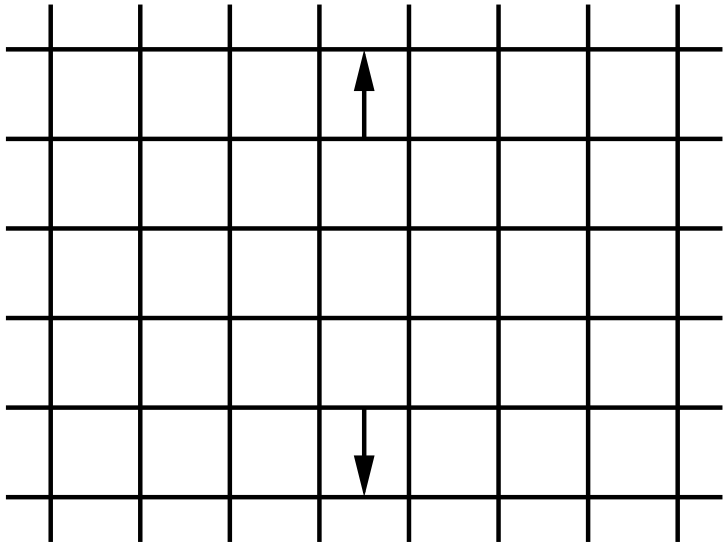

The local rule of HPP preserves the total number of particles and their total momentum. These are **conservation laws** that hold in the HPP automaton.

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(Note, however, that HPP is not a realistic lattice gas model as it satisfies incorrect conservation laws. For example, the horizontal momentum is preserved on each horizontal line individually.)

Even more interestingly, the HPP local rule fully preserves information. There is another CA that traces back the configurations in the reverse direction. This **inverse CA** simply moves the particles to the opposite direction, and applies the same collision rule as HPP.

HPP is an example of **reversible CA** (RCA), also called **invertible CA**. These are CA functions G such that there is another CA function F that is its inverse, i.e.

$$G \circ F = F \circ G = \text{identity function}.$$

RCA G and F are called the **inverse automata** of each other.

Clearly, in order to be reversible G has to be a bijective function $S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$.

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A CA is called

- **injective** if G is one-to-one,
- **surjective** if G is onto,
- **bijective** if G is both one-to-one and onto.

Outline of the talk(s)

(1) **Definitions** (configurations, neighborhoods, cellular automata, reversibility, injectivity, surjectivity)

(2) **Classical results** (Hedlund's theorem, balance in surjective CA, Garden-Of-Eden theorem)

(3) **Reversible CA** (universality, billiard ball CA)

(4) **Decidability questions** (Wang tiles, domino problem, snake tiles, undecidability of nilpotency, reversibility, surjectivity, periodicity)

(5) **Dynamical systems aspects** (equicontinuity classification, limit sets)

Classical results (1960's and 70's)

It is useful to endow the set $S^{\mathbb{Z}^d}$ of configurations with the **Cantor topology**, i.e. the topology generated by the **cylinder sets**

$$\mathrm{Cyl}(c, M) = \{e \in S^{\mathbb{Z}^d} \mid e(\vec{x}) = c(\vec{x}) \text{ for all } \vec{x} \in M\}$$

for $c \in S^{\mathbb{Z}^d}$ and finite $M \subset \mathbb{Z}^d$. The topology is compact, and it is induced by a metric.

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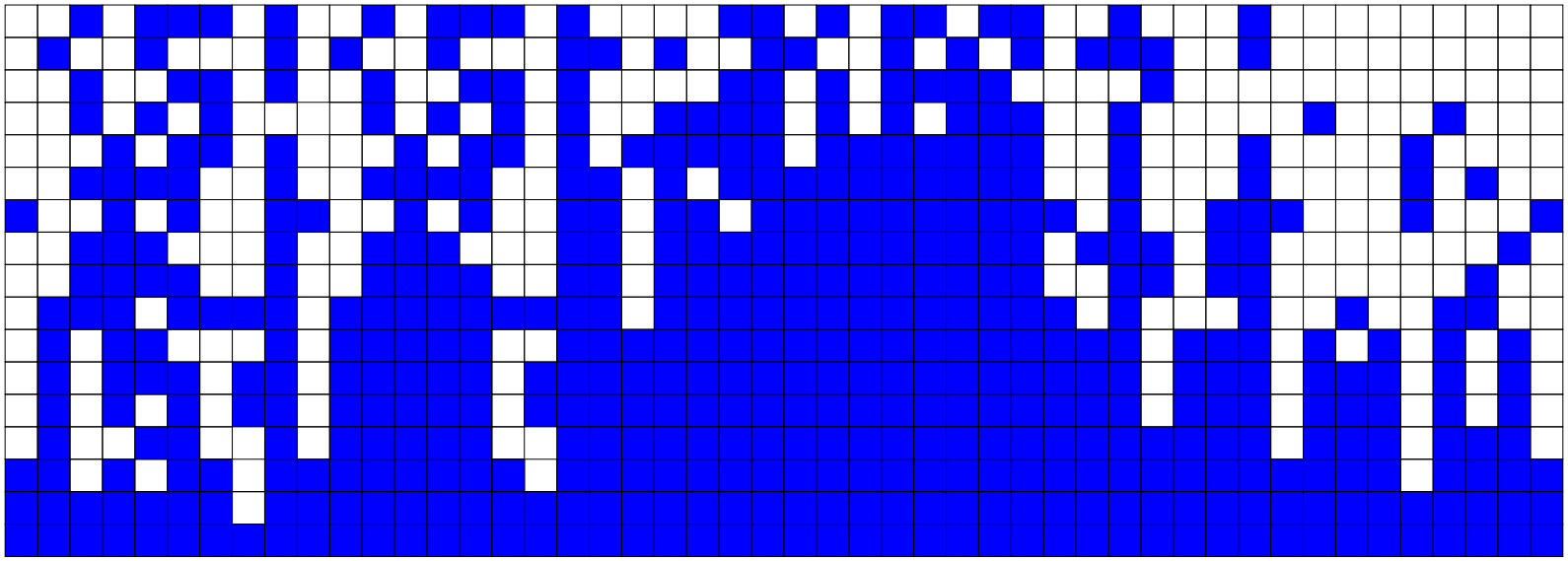
All cylinder sets are clopen, i.e. closed and open. Cylinders for fixed finite $M \subseteq \mathbb{Z}^d$ form a finite partitioning of $S^{\mathbb{Z}^d}$.

Under this topology, a sequence $c_1, c_2, \ldots$ of configurations **converges** to $c \in S^{\mathbb{Z}^d}$ if and only if for all cells $\vec{x} \in \mathbb{Z}^d$ and for all sufficiently large i holds

$$c_i(\vec{x}) = c(\vec{x}).$$



Compactness of the topology means that all infinite sequences $c_1, c_2, \ldots$ of configurations have converging subsequences.

All cellular automata are **continuous** transformations

$$S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$$

under the topology. Indeed, locality of the update rule means that if

$$c_1, c_2, \ldots$$

is a converging sequence of configurations then

$$G(c_1), G(c_2), \ldots$$

converges as well, and

$$\lim_{i \to \infty} G(c_i) = G(\lim_{i \to \infty} c_i).$$

The **translation** τ determined by vector $\vec{r} \in \mathbb{Z}^d$ is the transformation

$$S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$$

that maps $c \mapsto e$ where

$$e(\vec{x}) = c(\vec{x} - \vec{r}) \text{ for all } \vec{x} \in \mathbb{Z}^d.$$

(It is the CA whose local rule is the identity function and whose neighborhood consists of $-\vec{r}$ alone.)

Since all cells of a CA use the same local rule, the CA commutes with all translations:

$$G \circ \tau = \tau \circ G.$$

We have seen that all CA are continuous, translation commuting maps $S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$. The **Curtis-Hedlund-Lyndon theorem** from 1969 states also the converse:

Theorem: A function $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ is a CA function if and only if

(i) G is continuous, and

(ii) G commutes with translations.

Corollary: A cellular automaton G is reversible if and only if it is bijective.

Proof: If G is a reversible CA function then G is by definition bijective.

Conversely, suppose that G is a bijective CA function. Then G has an inverse function G^{-1} that clearly commutes with the shifts. The inverse function G^{-1} is also continuous because the space $S^{\mathbb{Z}^d}$ is compact. It now follows from the Curtis-Hedlund-Lyndon theorem that G^{-1} is a cellular automaton. $\square$

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The point of the corollary is that in bijective CA each cell can determine its previous state by looking at the current states in some bounded neighborhood around them.

Configurations that do not have a pre-image are called **Garden-Of-Eden** -configurations. A CA has GOE configurations if and only if it is non-surjective.

A finite pattern consists of a finite domain $M \subseteq \mathbb{Z}^d$ and an assignment

$$p : M \longrightarrow S$$

of states. Configuration c contains the pattern if

$$\tau(c)_{|M} = p$$

for some translation τ.

Finite pattern is called an **orphan** for CA G if every configuration containing the pattern is a GOE.

Let us show that every GOE contains an orphan:

Let c be a GOE, and let $M_1, M_2, M_3, \ldots$ be such that

- each $M_i \subseteq \mathbb{Z}^d$ is finite,
- $M_1 \subseteq M_2 \subseteq M_3 \subseteq \ldots$, and
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Since G is continuous also $G(S^{\mathbb{Z}^d})$ is compact. Hence sets

$$C_i = \mathrm{Cyl}(c, M_i) \cap G(S^{\mathbb{Z}^d})$$

are compact and form a decreasing chain $C_1 \supseteq C_2 \supseteq C_3 \supseteq \ldots$

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Because

$$C_1 \cap C_2 \cap C_3 \cap \ldots = \emptyset$$

it follows from compactness of that $C_i = \emptyset$ for some i. Then

$$c_{|M_i}$$

is an orphan. □

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$$\left|f^{-1}(a)\right| = |S|^{n-1}.$$

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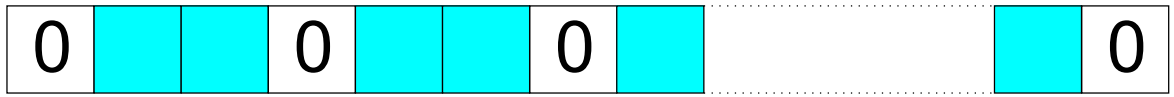
Indeed, consider a non-balanced local rule such as rule 110 where five contexts give new state 1 while only three contexts give state 0:

$$\begin{array}{ccc}
111 & \longrightarrow & 0 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 1 \\
010 & \longrightarrow & 1 \\
001 & \longrightarrow & 1 \\
000 & \longrightarrow & 0
\end{array}$$

Consider finite patterns where state 0 appears in every third position. There are $2^{2(k-1)} = 4^{k-1}$ such patterns where k is the number of 0's.



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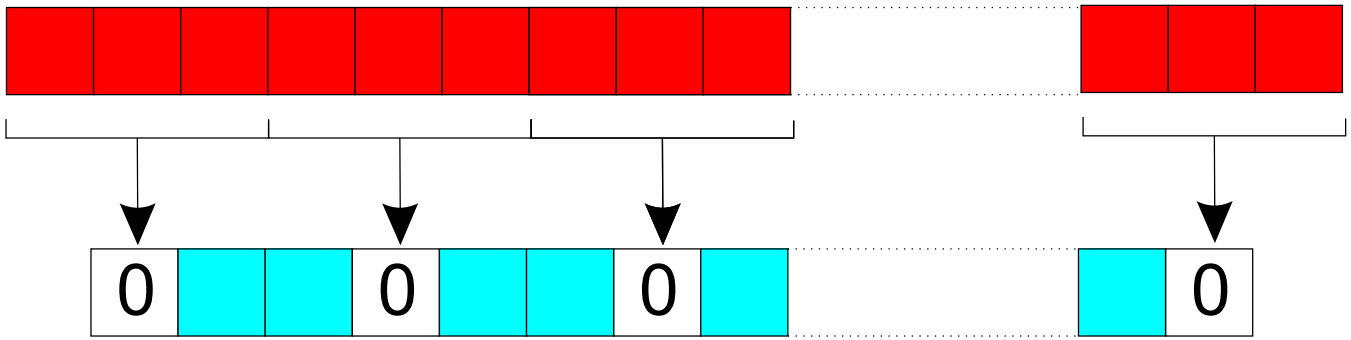

A pre-image of such a pattern must consist of k segments of length three, each of which is mapped to 0 by the local rule. There are 3^k choices.

As for large values of k we have $3^k < 4^{k-1}$, there are fewer choices for the red cells than for the blue ones. Hence some pattern has no pre-image and it must be an orphan.

□

One can also verify directly that pattern

$$01010$$

is an orphan of rule 110. It is the shortest orphan.

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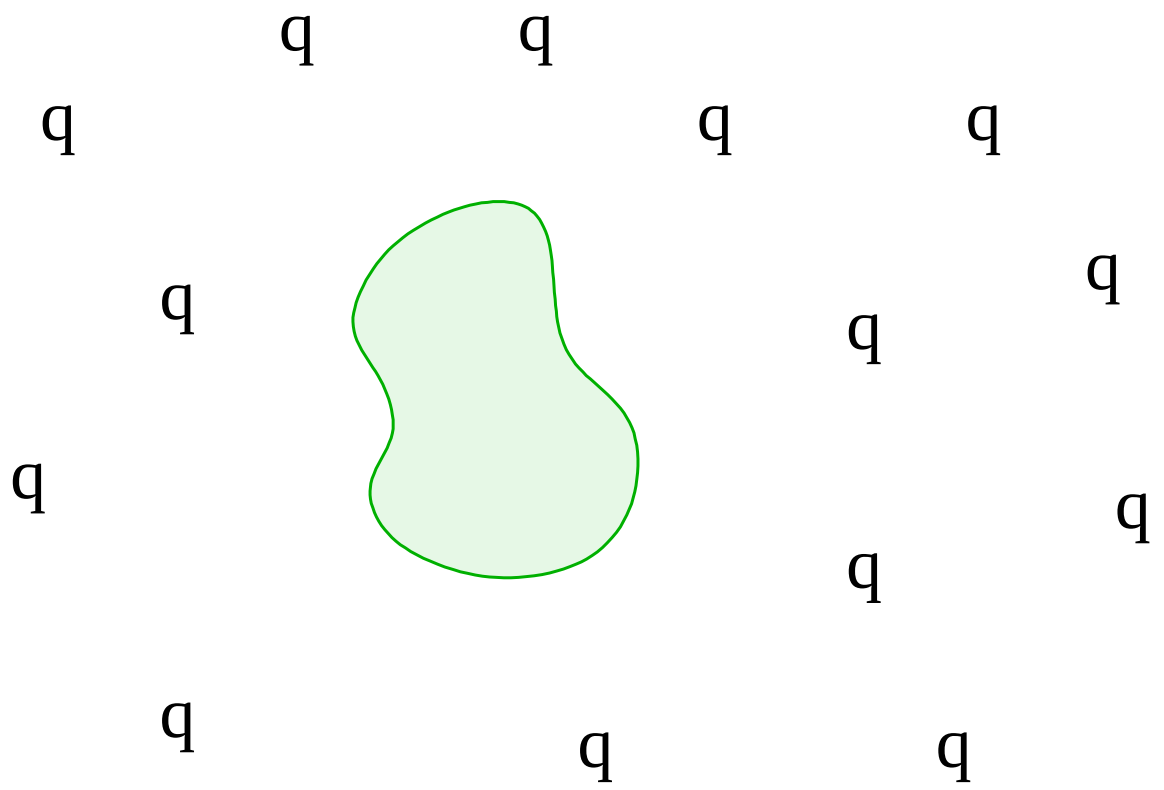
Balance of the local rule is not sufficient for surjectivity. For example, the **majority** CA (Wolfram number 232) is a counter example. The local rule

$$f(a, b, c) = 1 \text{ if and only if } a + b + c \geq 2$$

is clearly balanced, but 01001 is an orphan.

Sometimes one state $q \in S$ satisfying $f(q, q, \ldots, q) = q$ is specified as a **quiescent state**. A configuration is called **finite** if only a finite number of cells are non-quiescent:

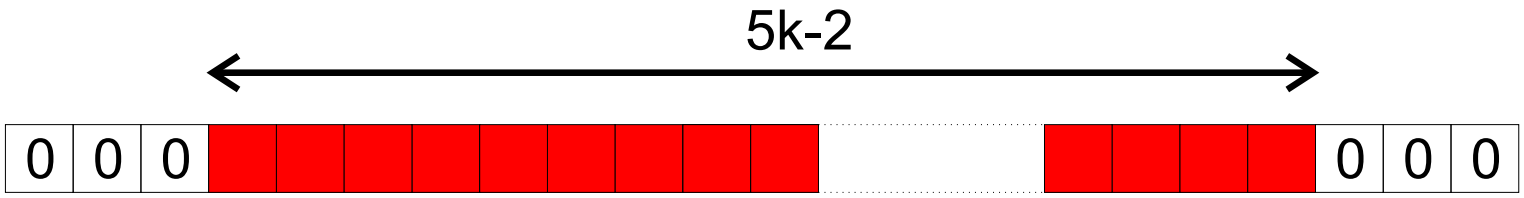


If c is finite and G is a CA function then also $G(c)$ is finite. Let us denote by G_F the restriction of G on the finite configurations.

The **Garden-Of-Eden -theorem** by Moore (1962) and Myhill (1963) states that G is surjective if and only if G_F is injective. Let us show this using rule 110 as a running example.

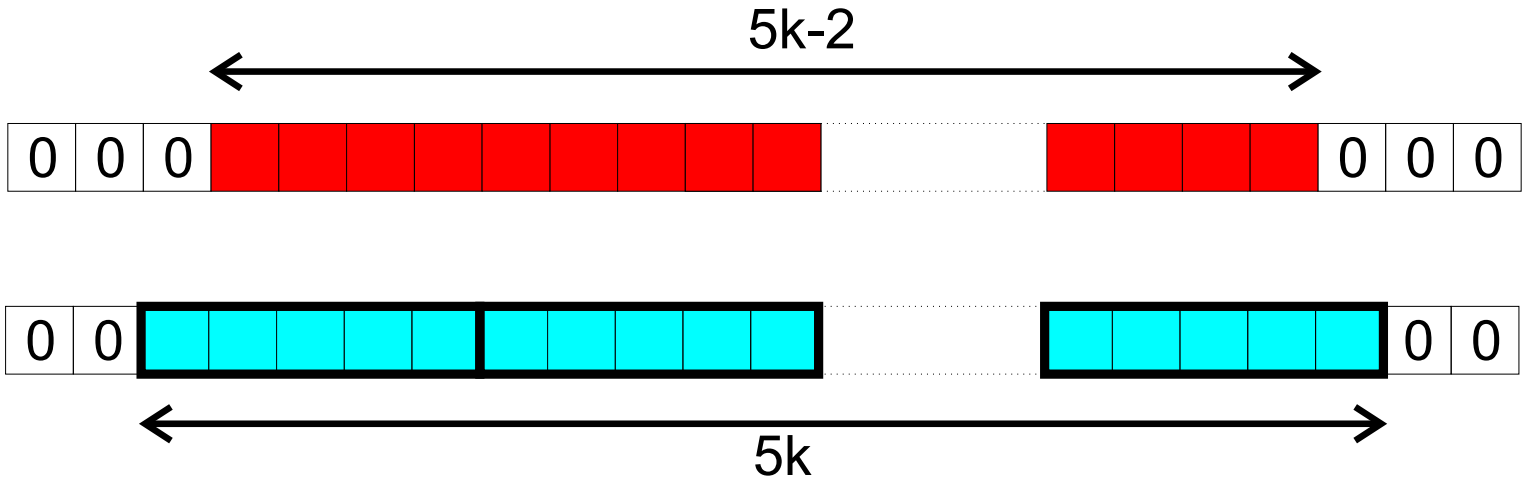
1) G not surjective $\Longrightarrow$ G_F not injective:

Since rule 110 is not surjective it has an orphan 01010 of width five. Consider a segment of length $5k-2$, for some k, and finite configurations c that are quiescent outside this segment. There are $2^{5k-2} = 32^k/4$ such configurations.



1) G not surjective $\Longrightarrow$ G_F not injective:

The non-quiescent part of $G(c)$ is within a segment of length $5k$. Partition this segment into k parts of length 5. Pattern 01010 cannot appear in any part, so only $2^5 - 1 = 31$ different patterns show up in the subsegments. There are at most 31^k possible configurations $G(c)$.



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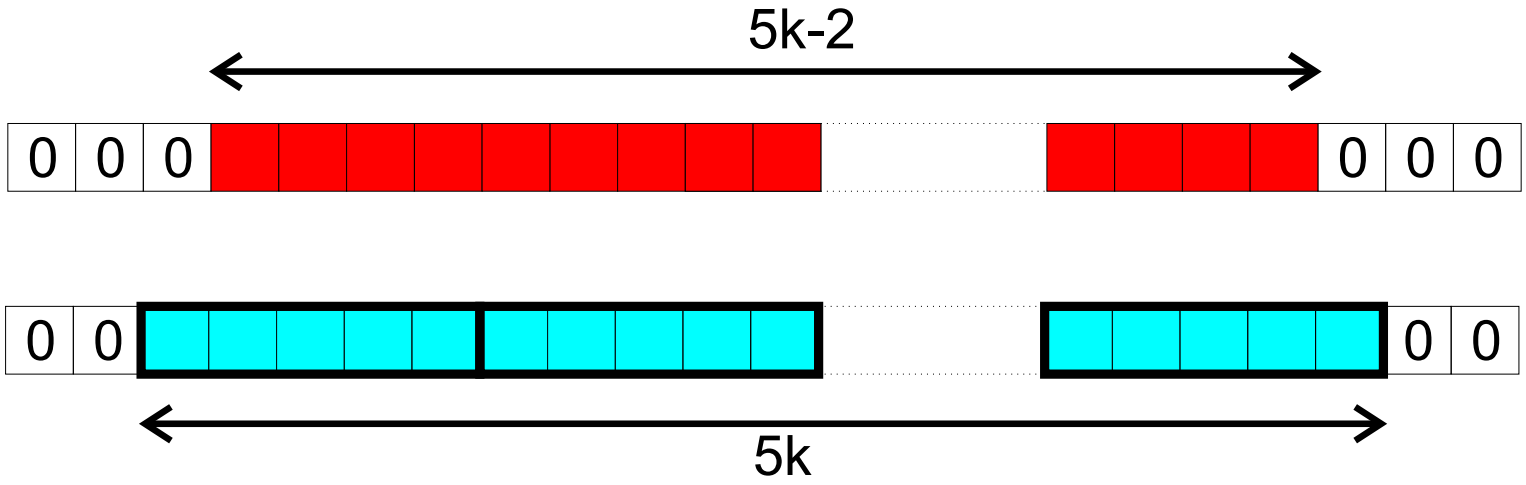
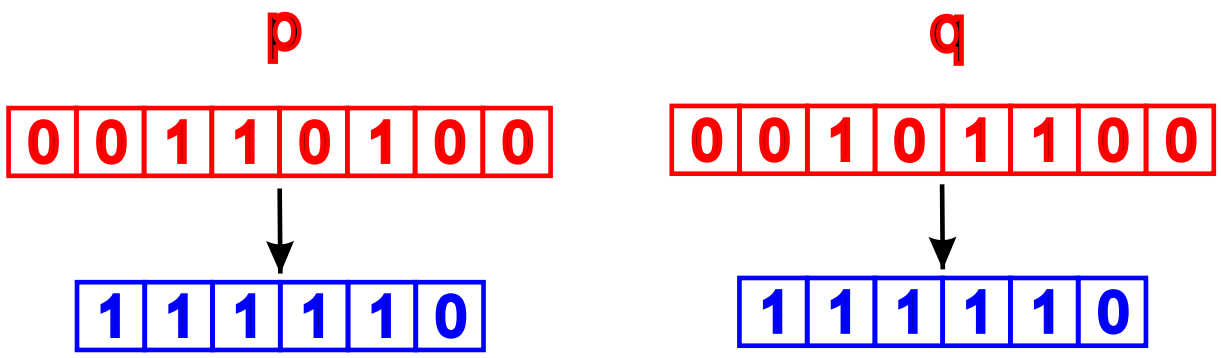


As $32^k/4 > 31^k$ for large k, there are more choices for red than blue segments. So there must exist two different red segments with the same image.

□

2) G_F not injective $\Longrightarrow$ G not surjective:

In rule 110



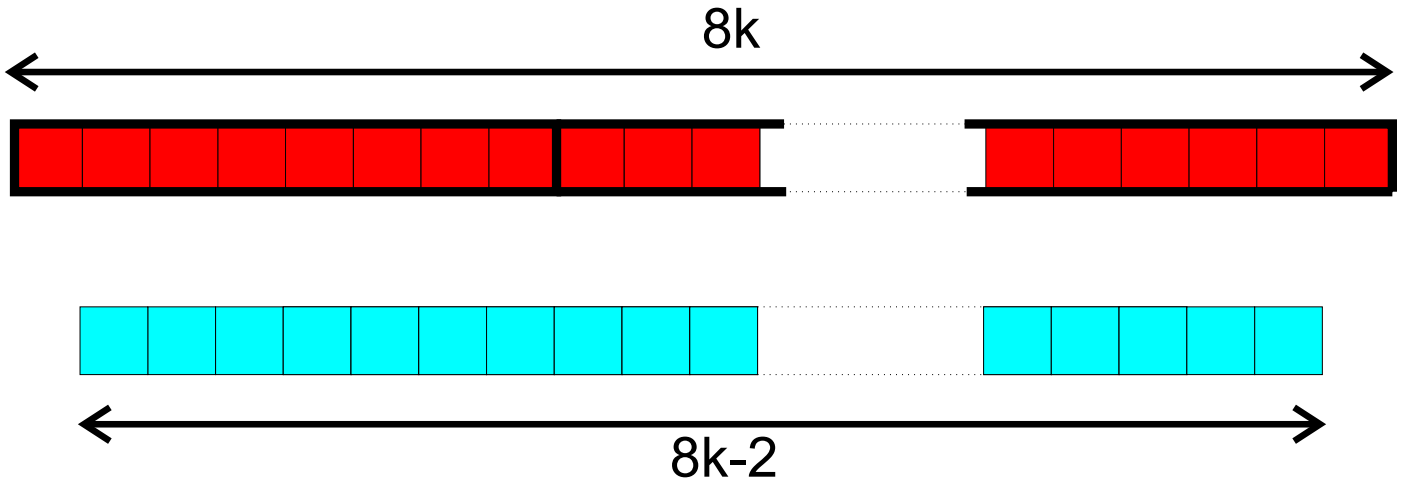
so patterns p and q of length 8 can be exchanged to each other in any configuration without affecting its image. There exist just

$$2^8 - 1 = 255$$

essentially different blocks of length 8.

2) G_F not injective $\Longrightarrow$ G not surjective:

Consider a segment of $8k$ cells, consisting of k parts of length 8. Patterns p and q are exchangeable, so the segment has at most 255^k different images.



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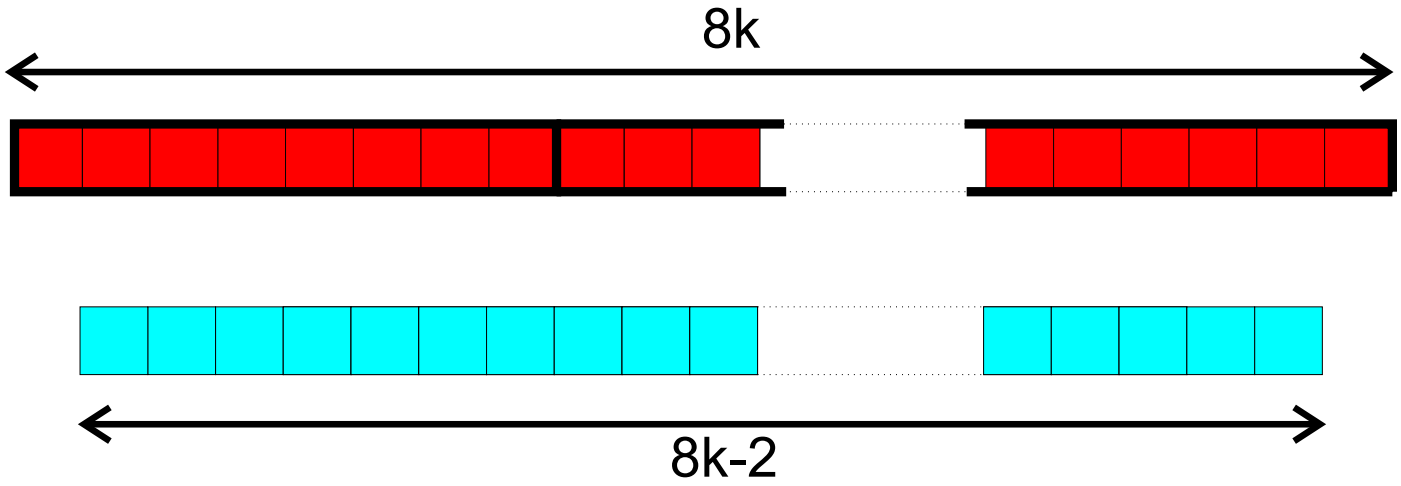

There are, however, $2^{8k-2} = 256^k/4$ different patterns of size $8k-2$. Because $255^k < 256^k/4$ for large k, there are blue patterns without any pre-image.

□

Theorem: CA G is surjective if and only if its restriction G_F to finite configurations is injective. □

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Proof: If G is injective then also G_F is injective. The claim follows from the Garden-Of-Eden -theorem. □

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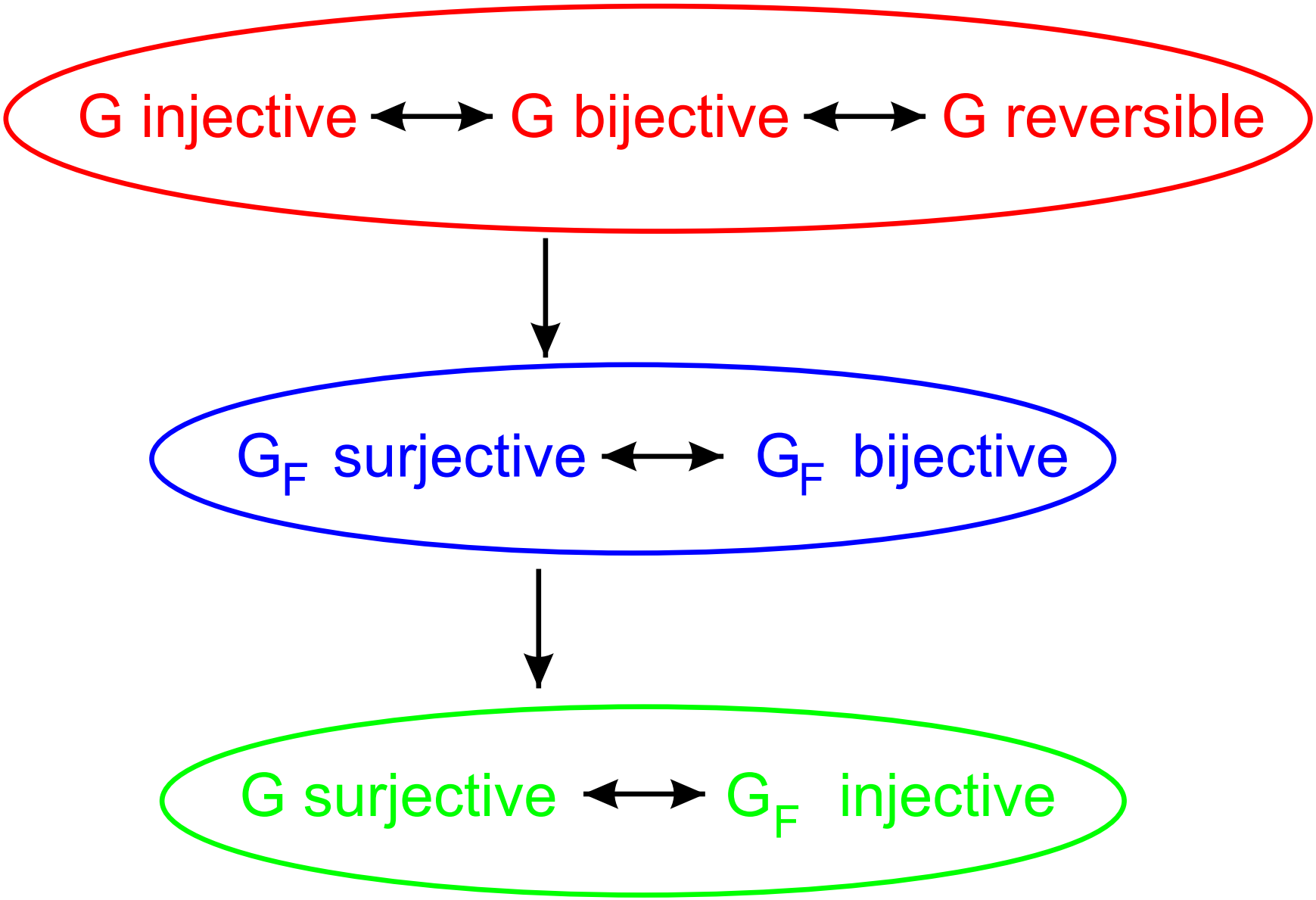
Corollary: If G is injective then G_F is surjective.

Proof: If G is injective then it is reversible. The quiescent state of G is also quiescent in G^{-1}, so G^{-1} maps finite configurations to finite configurations. So every finite configuration has a finite pre-image. □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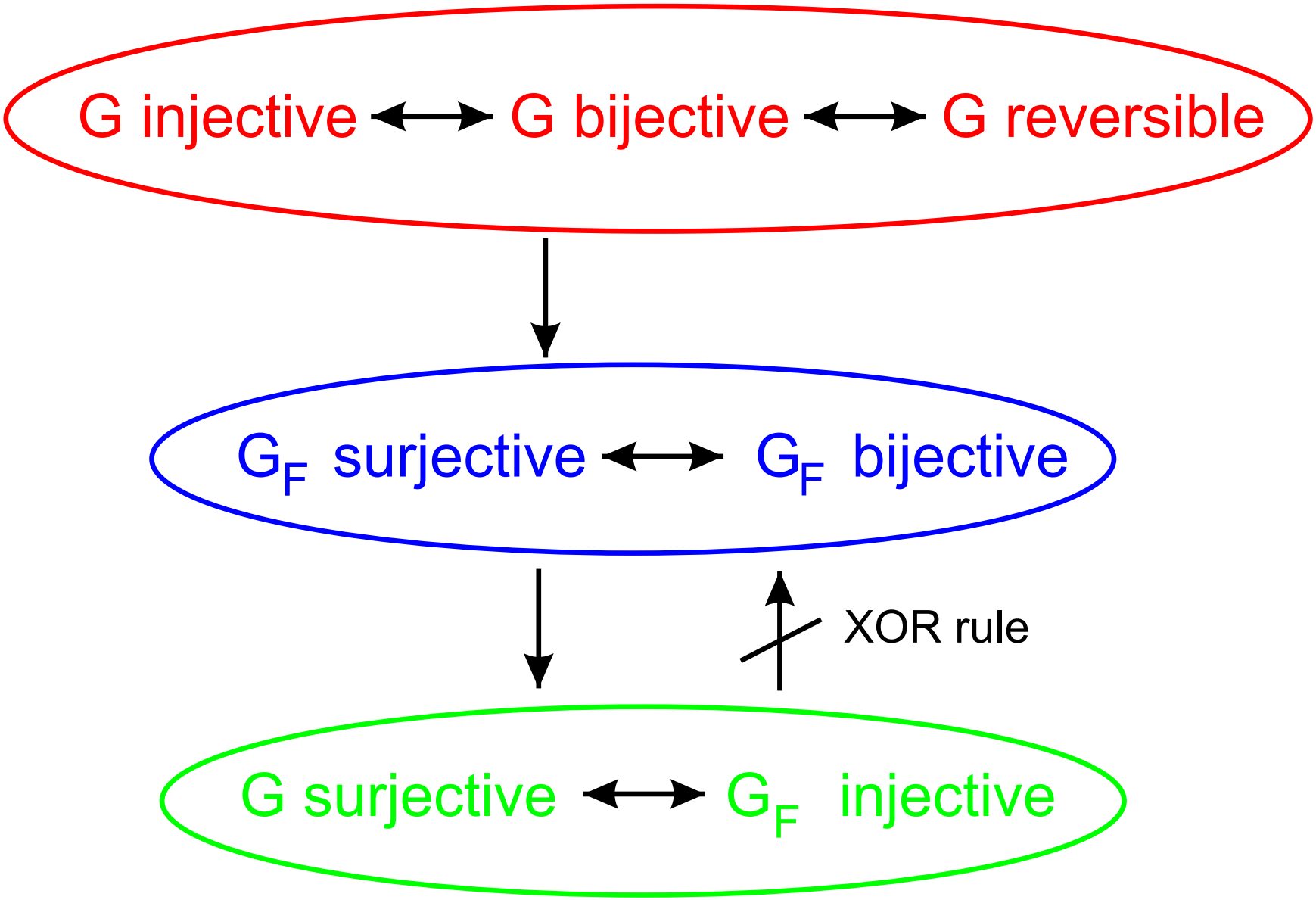
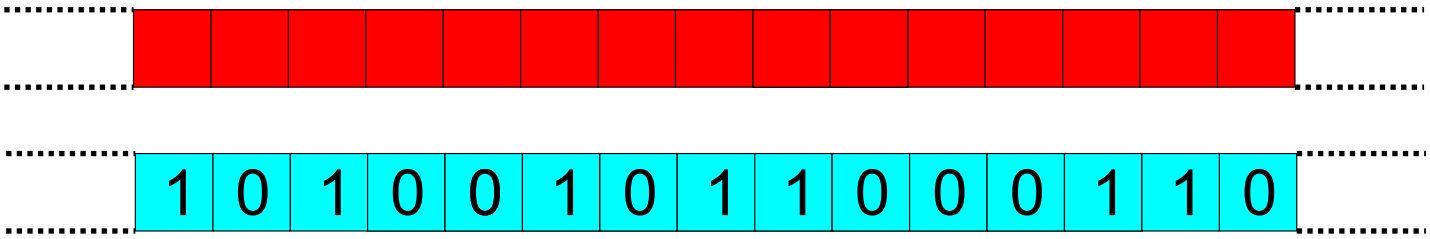
G_F surjective ⟷ G_F bijective
G surjective ⟷ G_F injective


G injective
G bijective
G reversible
G_F surjective
G_F bijective
XOR rule
G surjective
G_F injective

In the xor-CA every configuration has exactly two pre-images, so G is surjective:



One can freely choose one value in the pre-image, after which all remaining states are uniquely determined by the **left-permutativity** and the **right-permutativity** of xor.

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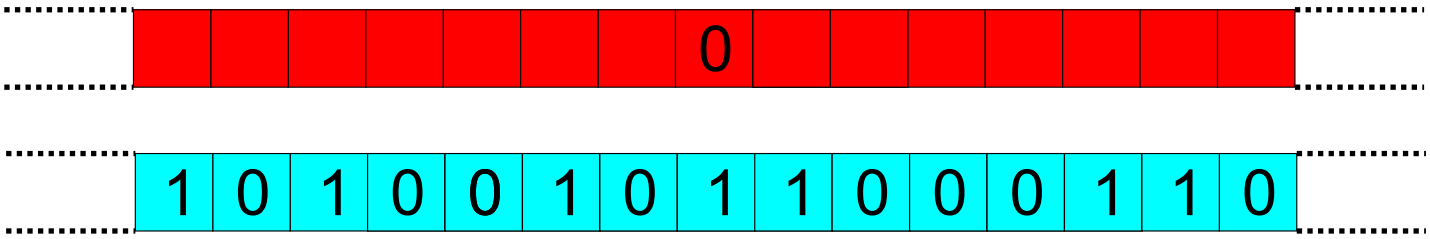

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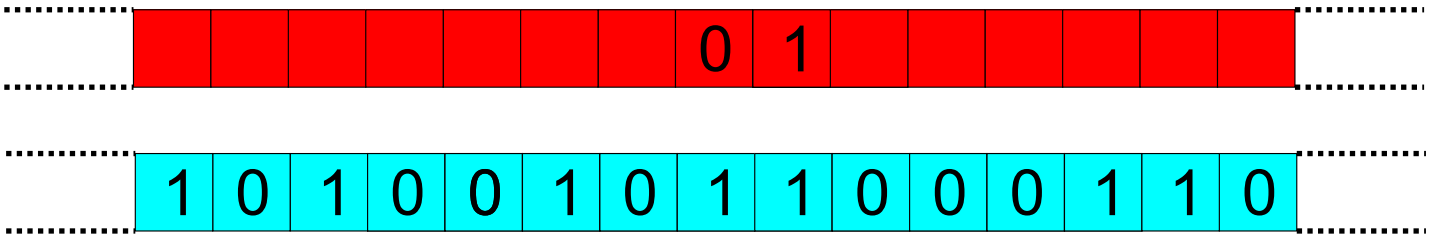

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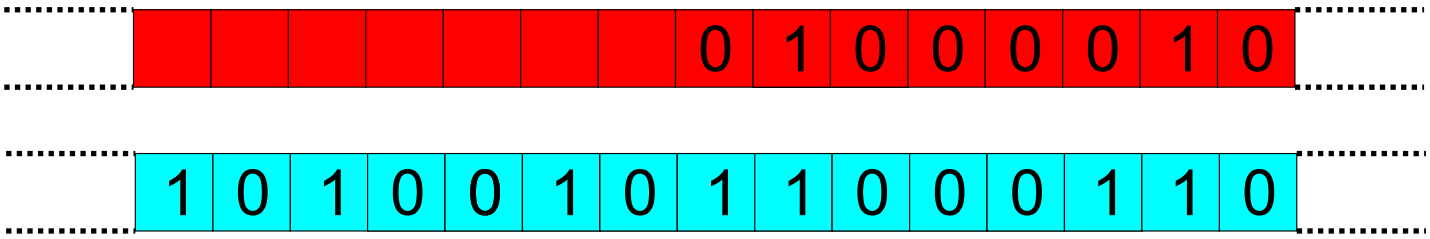

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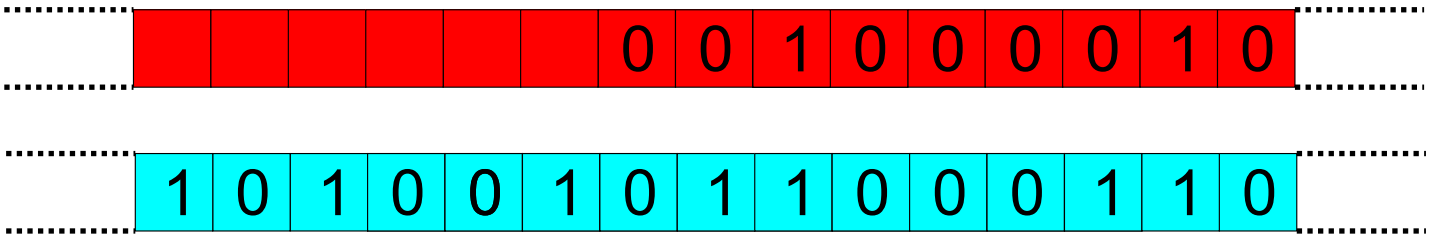

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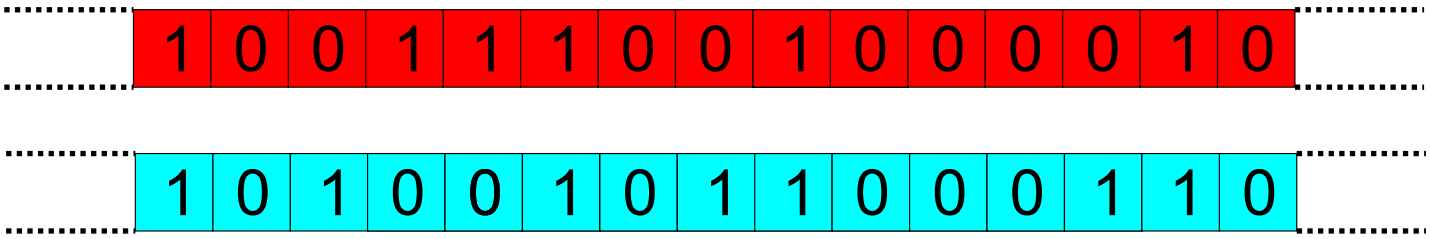

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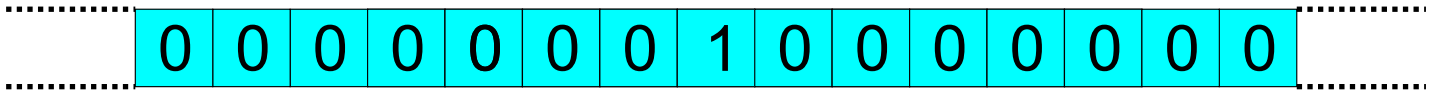
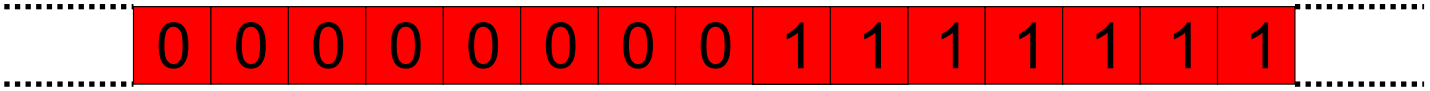
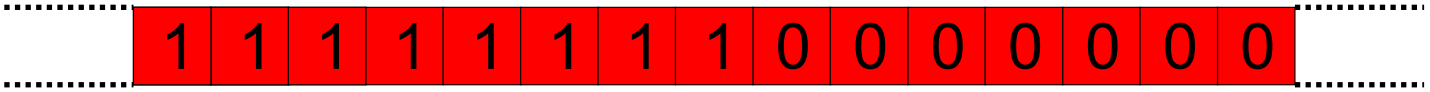
The two pre-images of the finite configuration

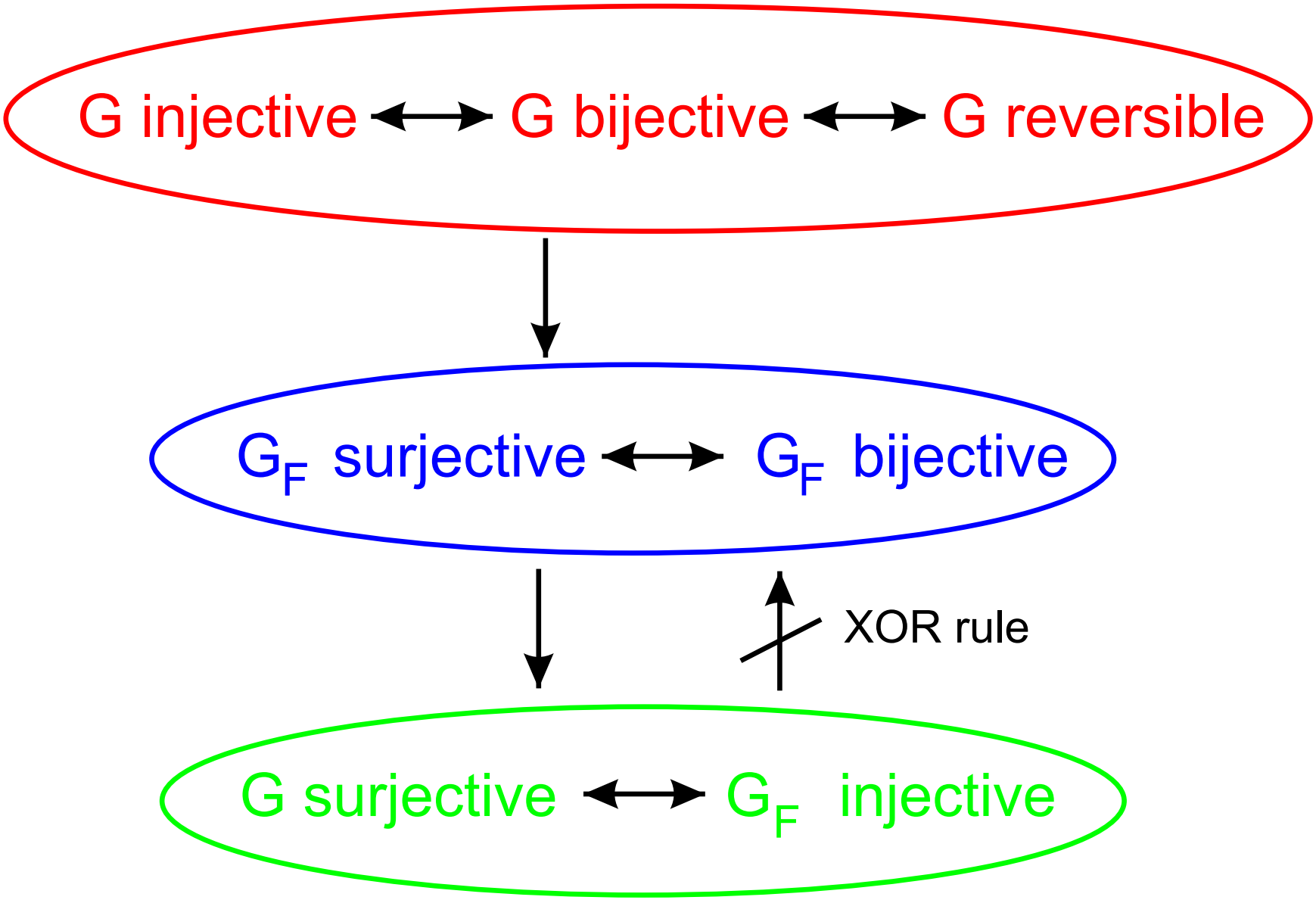


are both infinite:





So G_F is not surjective.


G injective
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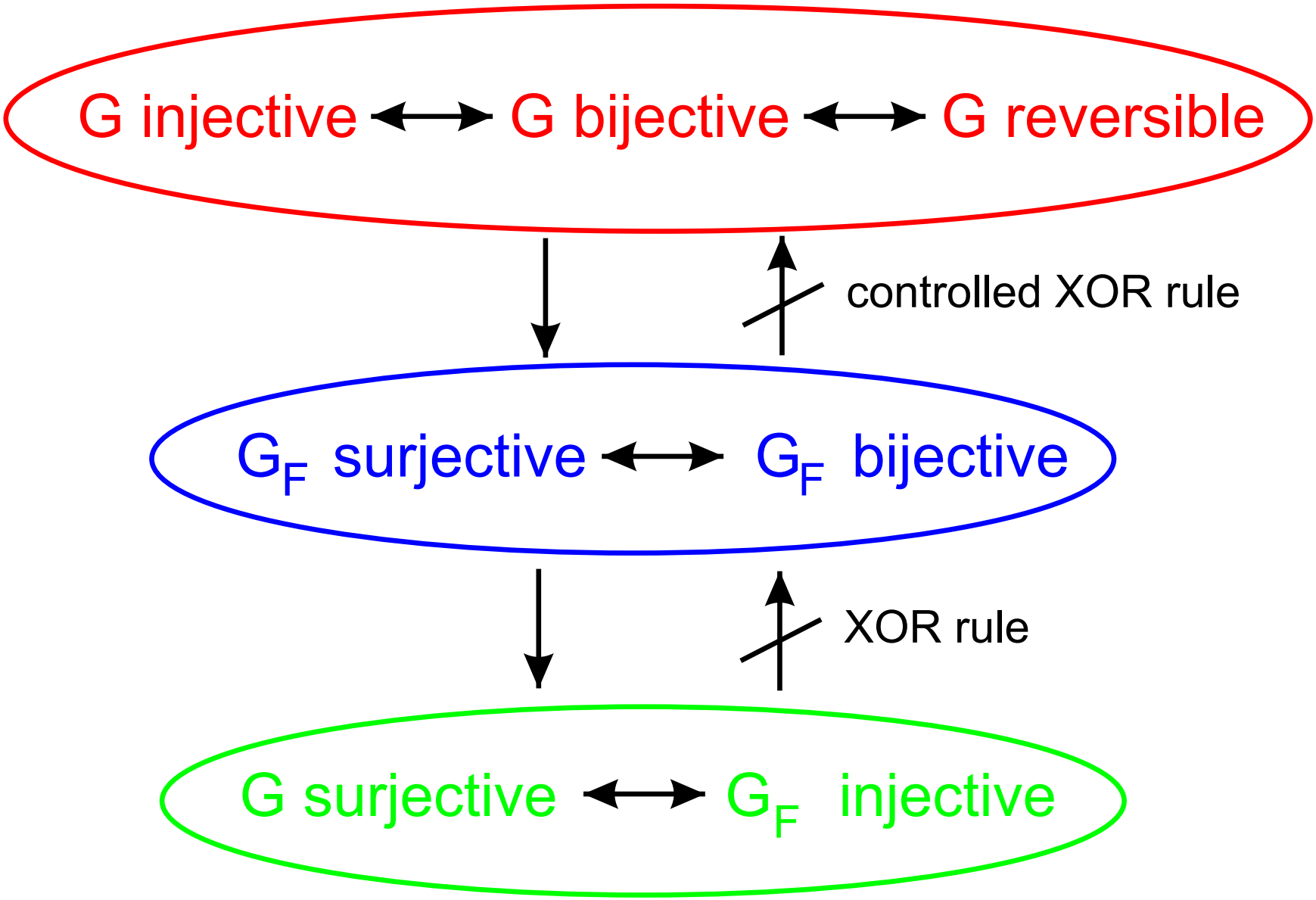
G_F injective


G injective
G bijective
G reversible
controlled XOR rule
G_F surjective
G_F bijective
XOR rule
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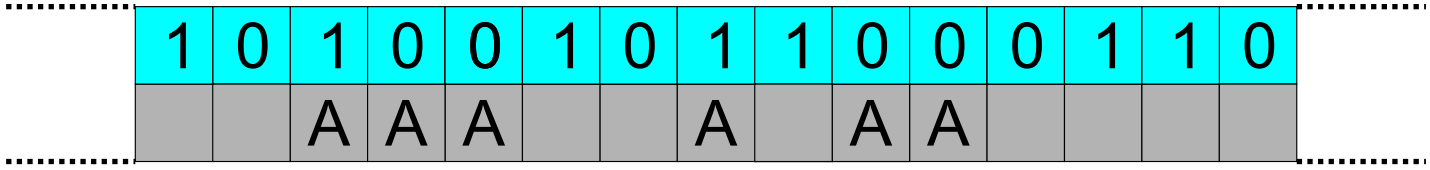
The **controlled-xor** CA has two binary tracks. The lower track does not change, but it indicates which cells are active. Non-active cells do not change at all, while active cells apply the xor rule on the first track:



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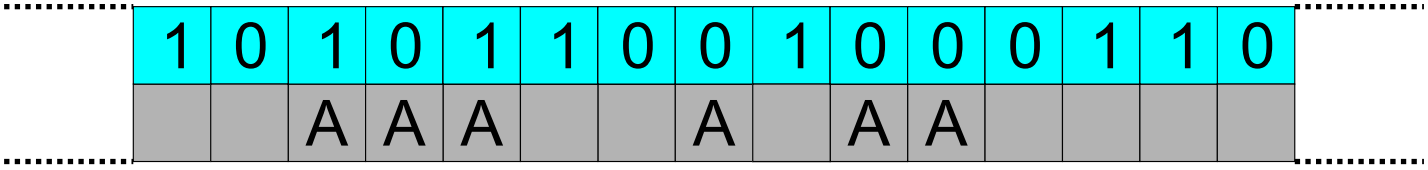

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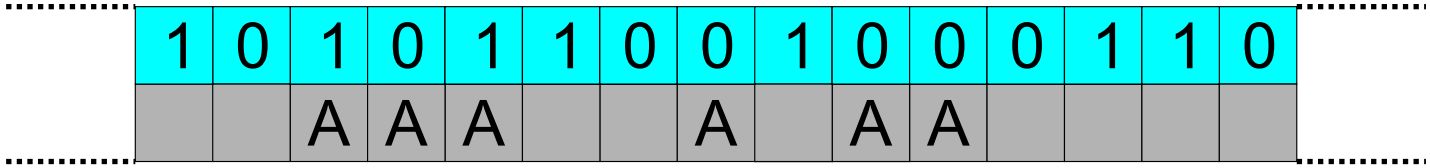
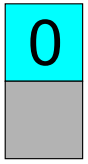


The quiescent state is 

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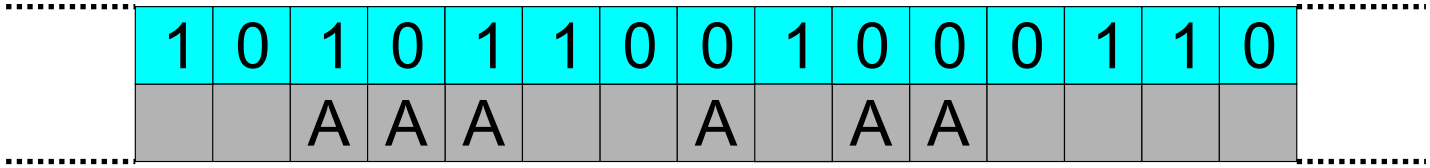
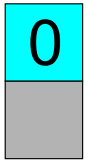
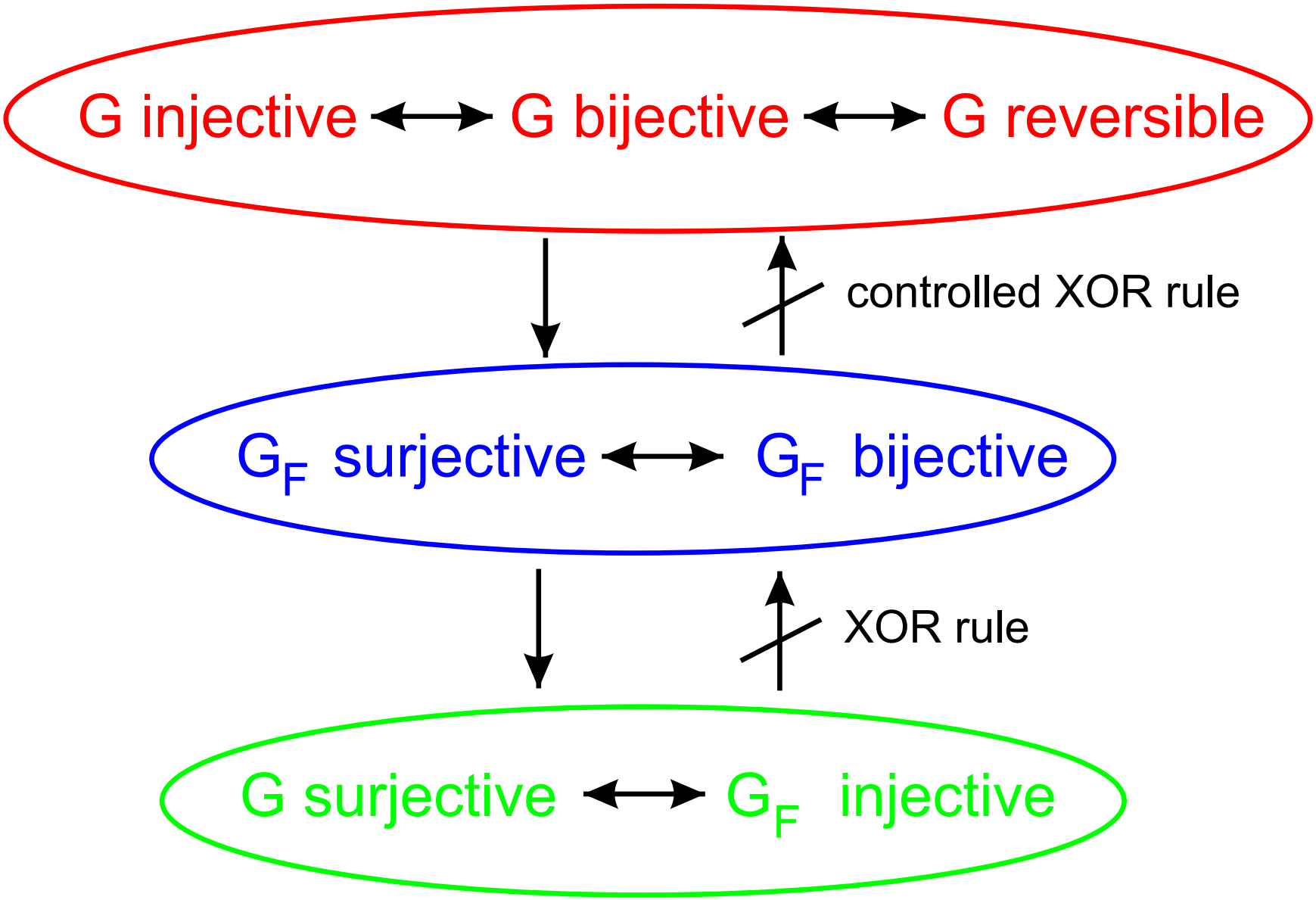


The quiescent state is 

The CA is not injective as all active 0's and all active 1's have the same image, but every finite configuration has a finite pre-image, so G_F is surjective.


G injective
G bijective
G reversible
controlled XOR rule
G_F surjective
G_F bijective
XOR rule
G surjective
G_F injective

Examples:

The majority rule is not surjective: finite configurations

$$\ldots 0000000 \ldots \qquad \text{and} \qquad \ldots 0001000 \ldots$$

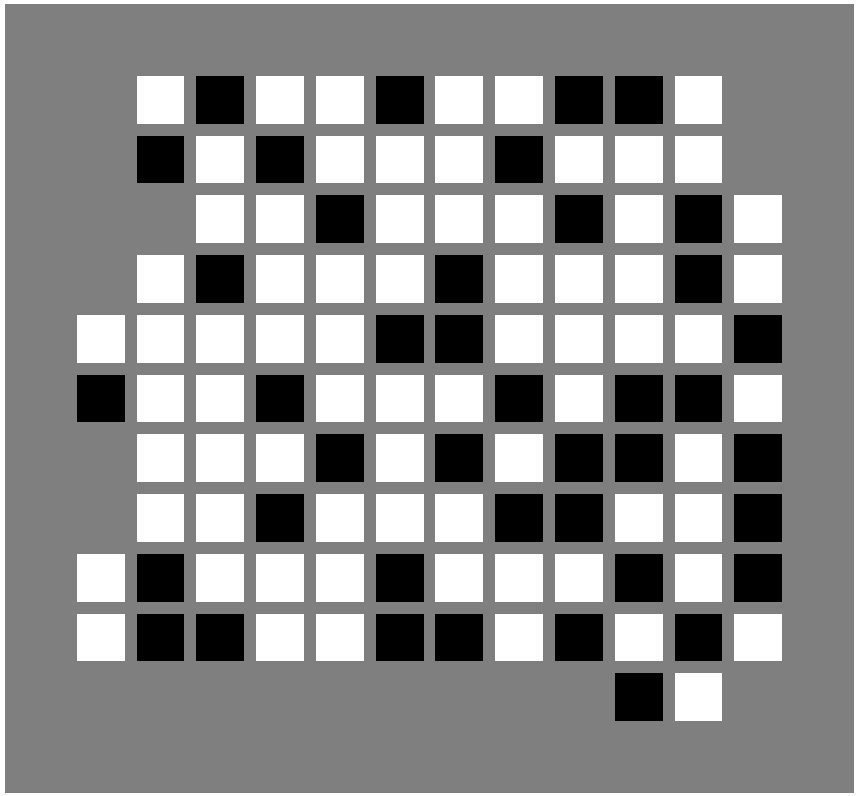
have the same image, so G_F is not injective. Pattern

$$01001$$

is an orphan.

Examples:

In Game-Of-Life a lonely living cell dies immediately, so G_F is not injective. So GOL is not surjective. Interestingly, no small orphans are known for Game-Of-Life. Currently, the smallest known orphan consists of 113 cells:



Examples:

The **Traffic CA** is the elementary CA number 226.

$$
\begin{array}{lcl}
111 & \longrightarrow & 1 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
100 & \longrightarrow & 0 \\
011 & \longrightarrow & 0 \\
010 & \longrightarrow & 0 \\
001 & \longrightarrow & 1 \\
000 & \longrightarrow & 0
\end{array}
$$

The local rule replaces pattern 01 by pattern 10.

$$
\begin{array}{ccc}
111 & \longrightarrow & 1 \\
110 & \longrightarrow & 1 \\
101 & \longrightarrow & 1 \\
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011 & \longrightarrow & 0 \\
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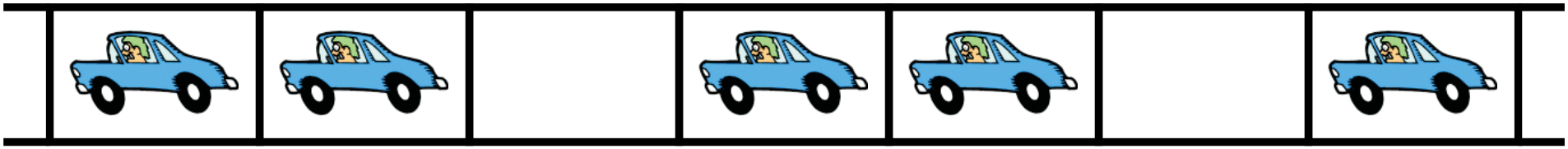
\end{array}
$$



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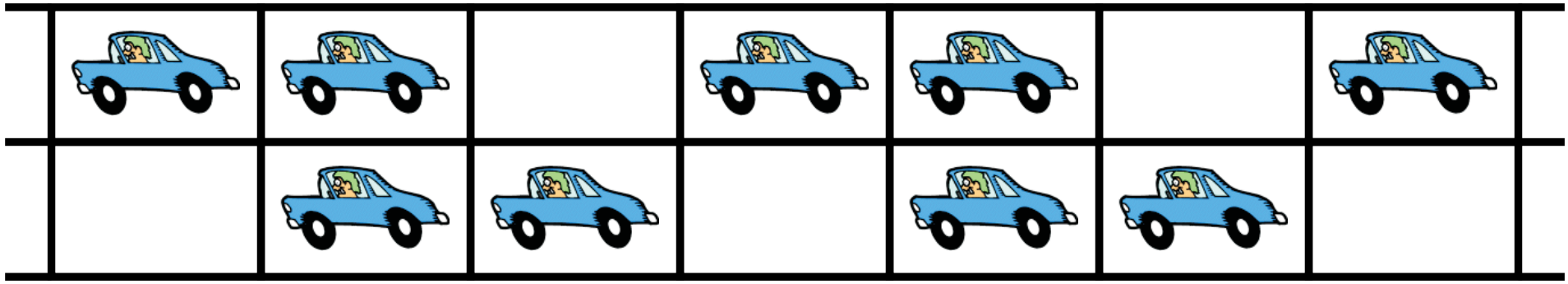

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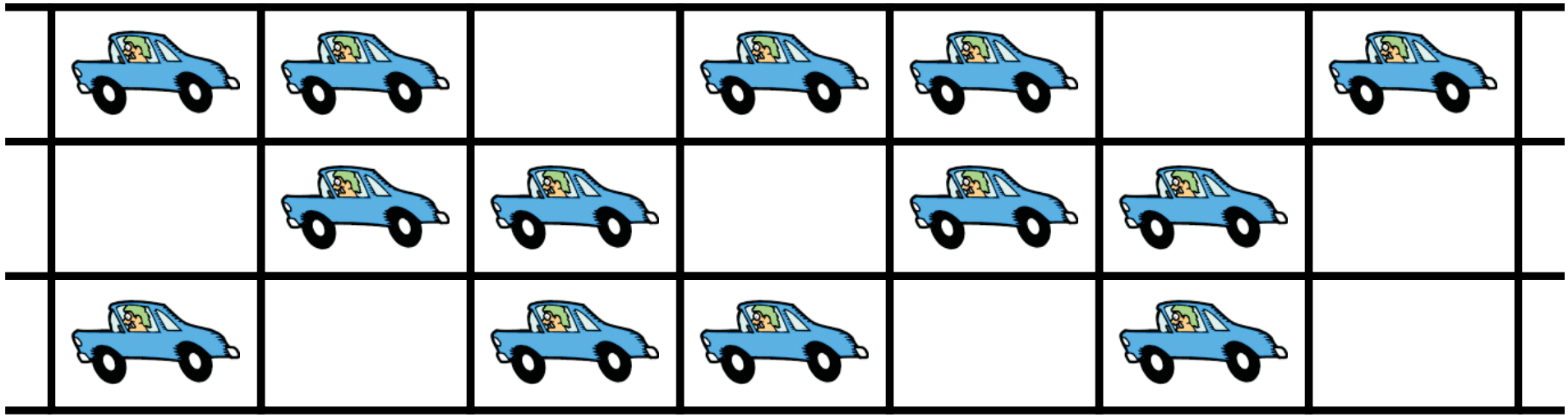

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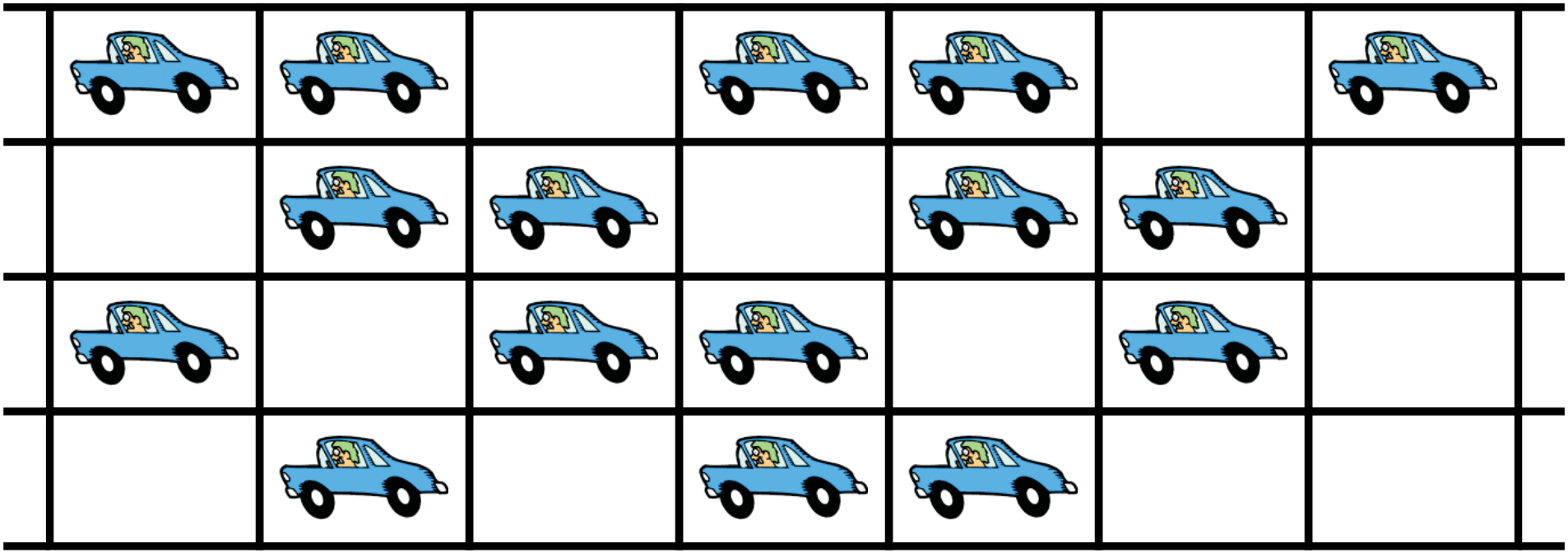
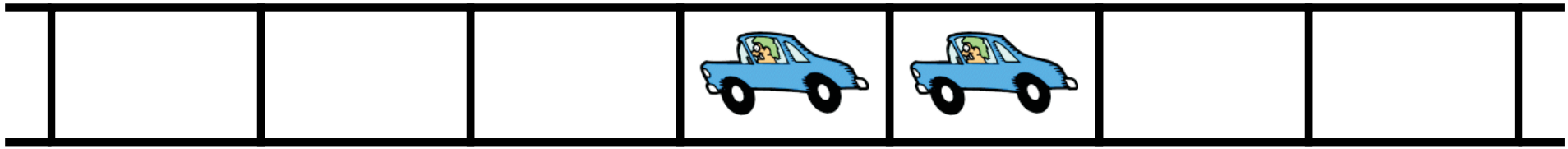
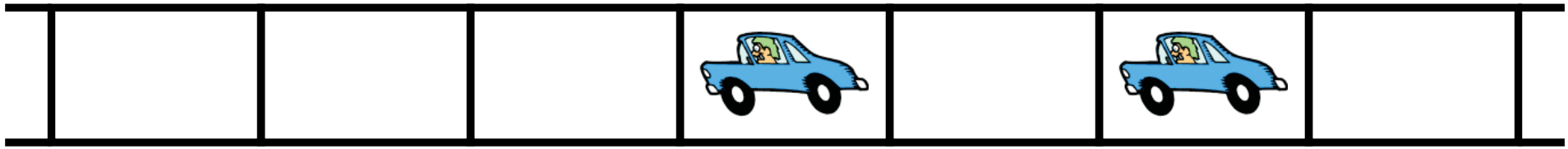


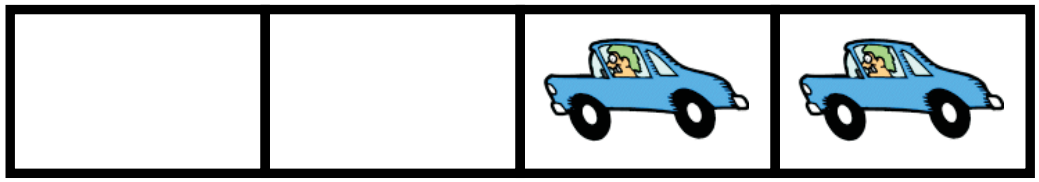
The local rule is balanced. However, there are two finite configurations with the same successor:





and hence traffic CA is not surjective.

There is an orphan of size four:



Outline of the talk(s)

(1) **Definitions** (configurations, neighborhoods, cellular automata, reversibility, injectivity, surjectivity)

(2) **Classical results** (Hedlund's theorem, balance in surjective CA, Garden-Of-Eden theorem)

(3) **Reversible CA** (universality, billiard ball CA)

(4) **Decidability questions** (Wang tiles, domino problem, snake tiles, undecidability of nilpotency, reversibility, surjectivity, periodicity)

(5) **Dynamical systems aspects** (equicontinuity classification, limit sets)

Reversible cellular automata (RCA)

Reversibility is an important aspect of physics at the microscopic scale, so it is natural that simulations are done using reversible CA. Despite being a very restrictive condition, reversibility does not prevent universal computation.

It is clear that any Turing machine can be simulated by a one-dimensional CA. In 1977 T.Toffoli demonstrated how any d-dimensional CA can be simulated by a $d+1$-dimensional RCA. Hence two-dimensional universal RCA exist.

In 1989 K.Morita and M.Harao showed how to simulate any reversible Turing machine by a one-dimensional RCA. Since reversible Turing machines can be computation universal (Bennett 1973), the following result follows:

Theorem: One-dimensional reversible cellular automata exist that are computationally universal.

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Theorem: One-dimensional reversible cellular automata exist that are computationally universal.

In the following I give a simple construction of a one-dimensional reversible CA that simulates in real time any Turing machine, reversible or not.

Partitioned CA (PCA) are particular types of reversible cellular automata. The state set of a PCA is a cartesian product of finite sets:

$$S = S_1 \times S_2 \times \ldots S_k.$$

The k components are called **tracks**. The local update rule is specified by providing

- a translation τ_i for each track $i = 1, 2, \ldots, k$, and
- a bijective transformation

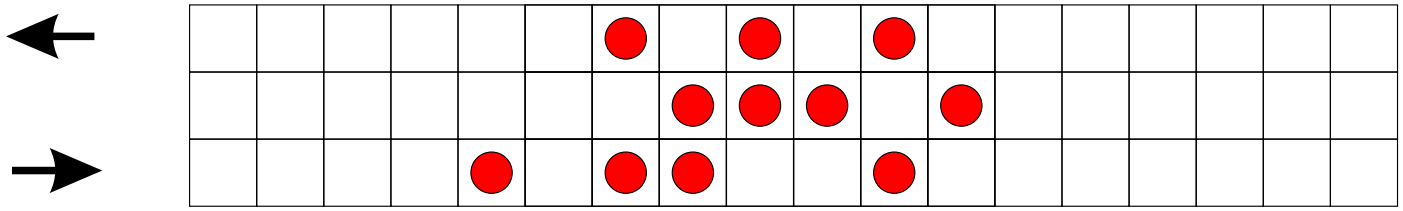
$$\pi : S \longrightarrow S$$

  of the entire state set.

The update is done in two steps: First π is applied at all cells, after which all tracks are translated according to τ_i.

Example: A three track PCA where each track is binary. Translations of tracks are (1) shift left, (2) no move and (3) shift right. Permutation π is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \quad \mapsto \quad \begin{bmatrix} a & \\ a+b & \pmod 2 \\ b+c & \pmod 2 \end{bmatrix}$$



PCA are clearly reversible as both elementary steps are reversible.

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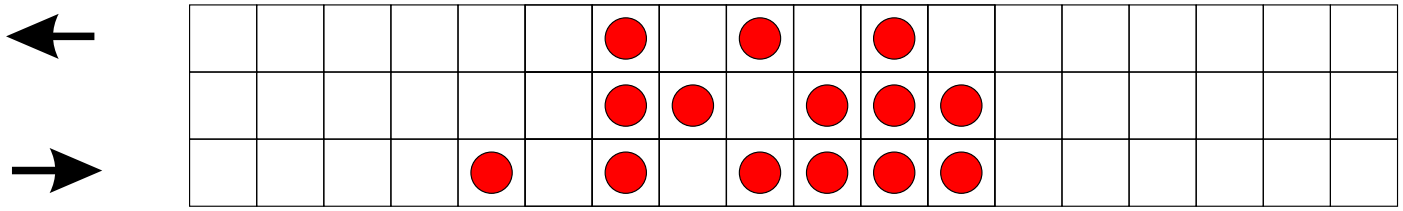

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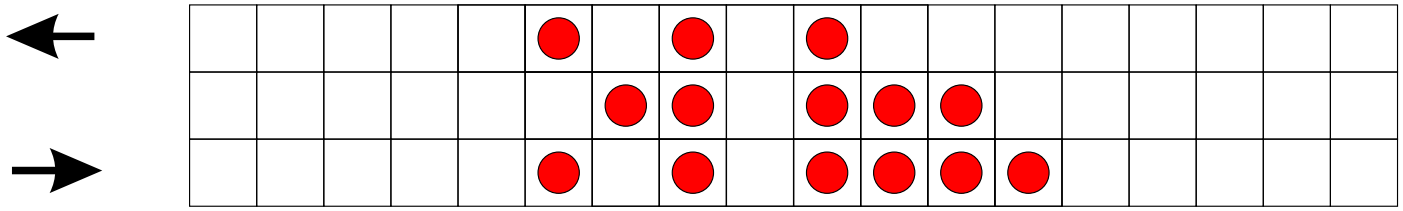

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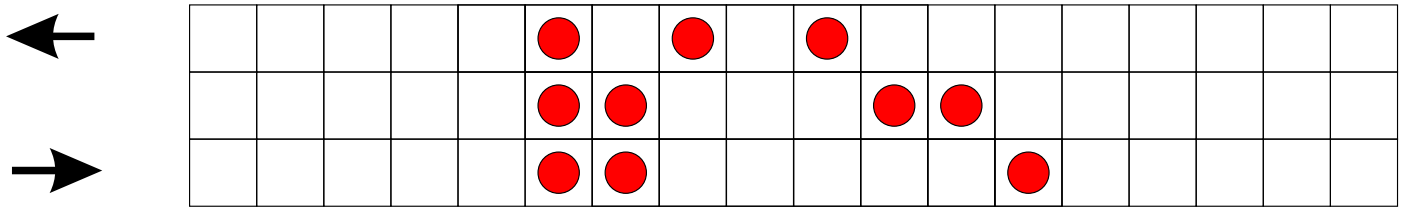

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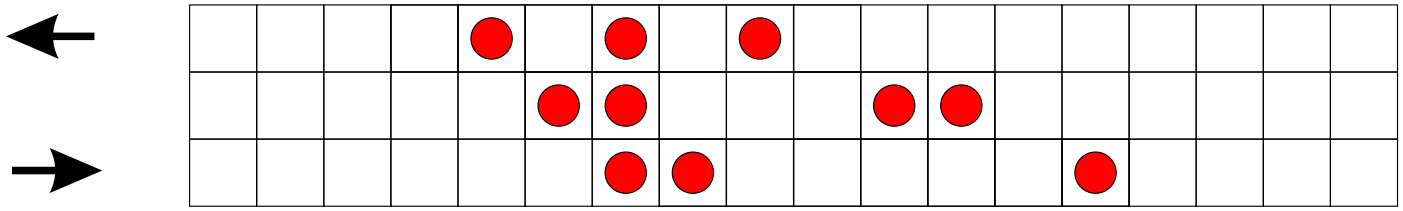

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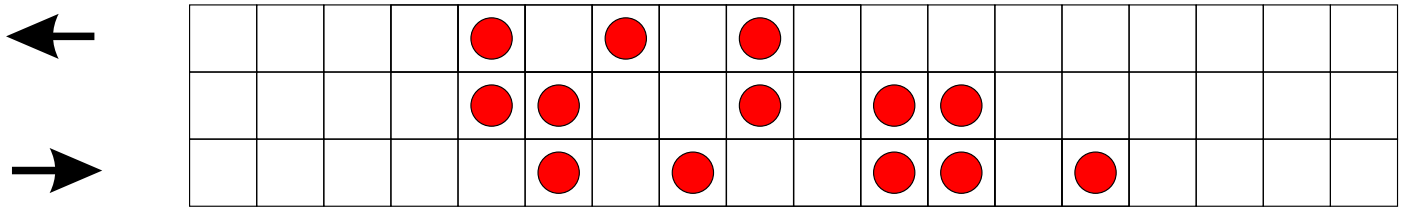

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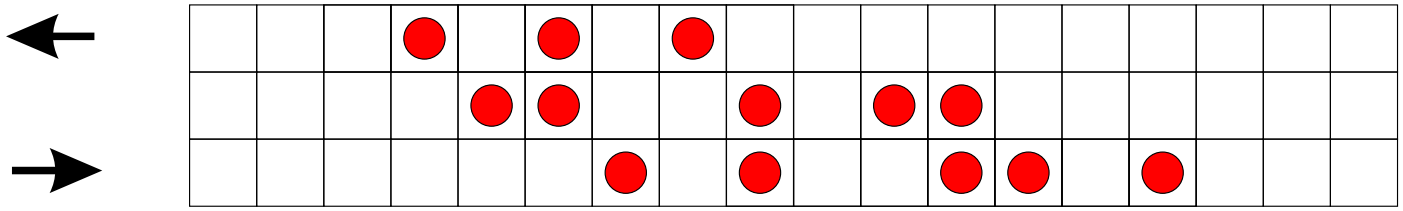

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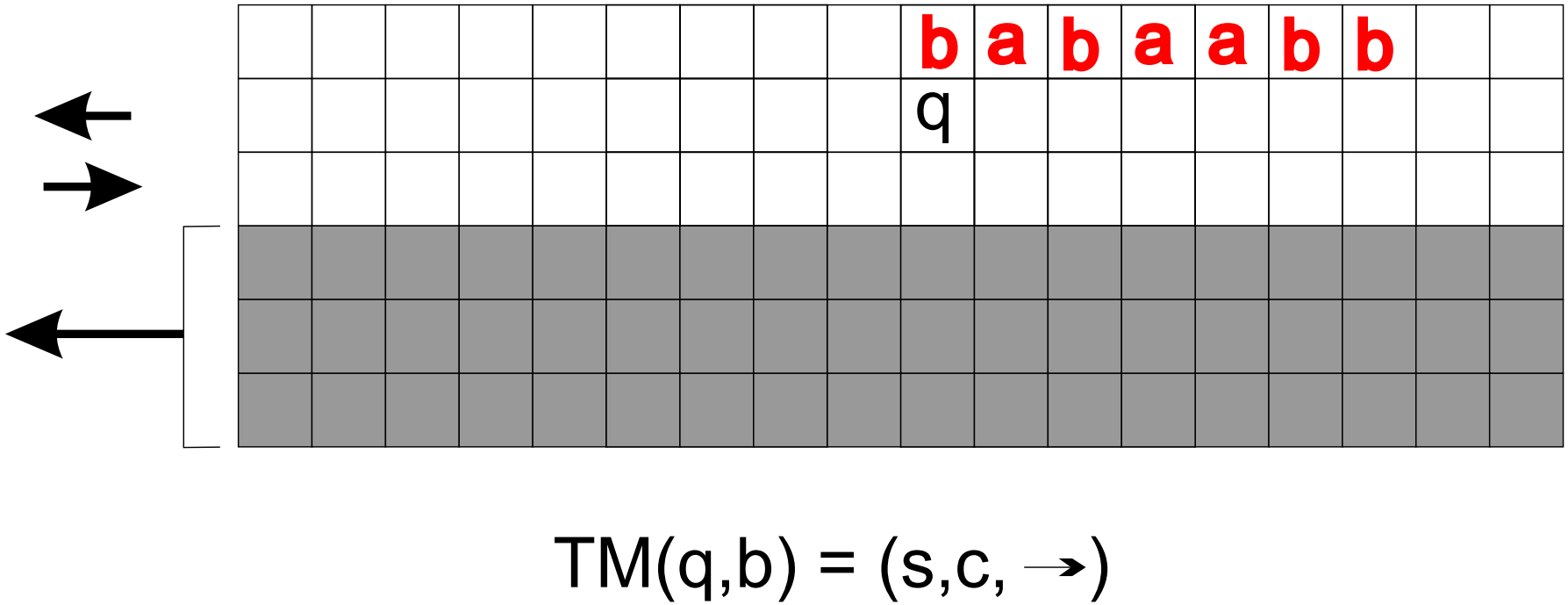
Track (4) is a garbage track. It is translated by two cells so that a new empty "trash bin" always appears at the position of the Turing machine.

The permutation π copies first the contents of tracks (1),(2) and (3) to the garbage track. This copying is done only at the cell that contains the Turing machine head. Then the first three tracks are updated according to the Turing machine instruction.



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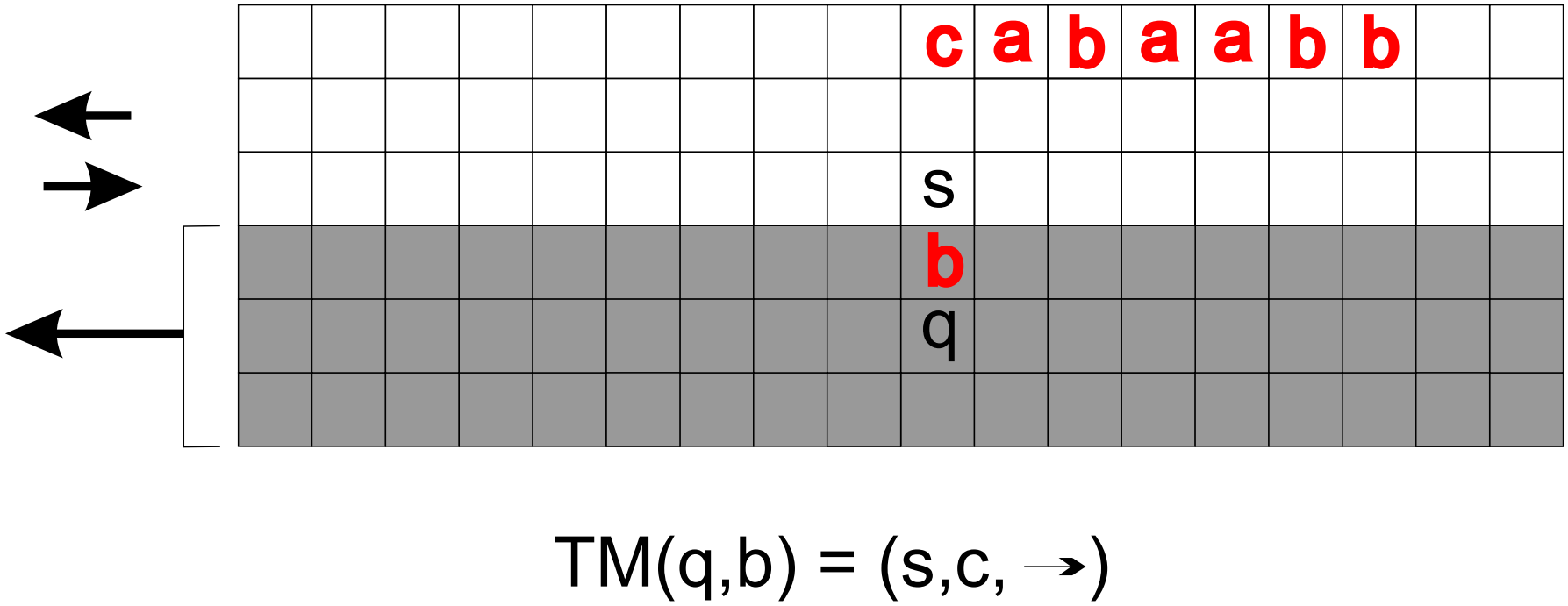

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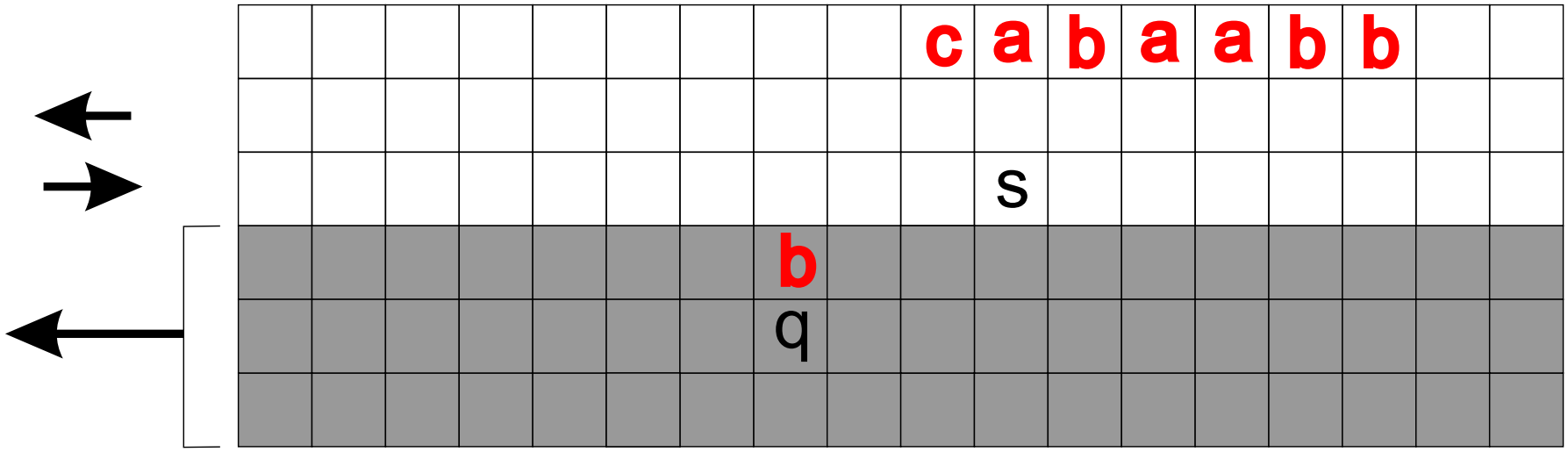

TM(s,a) = (r,d, ←)

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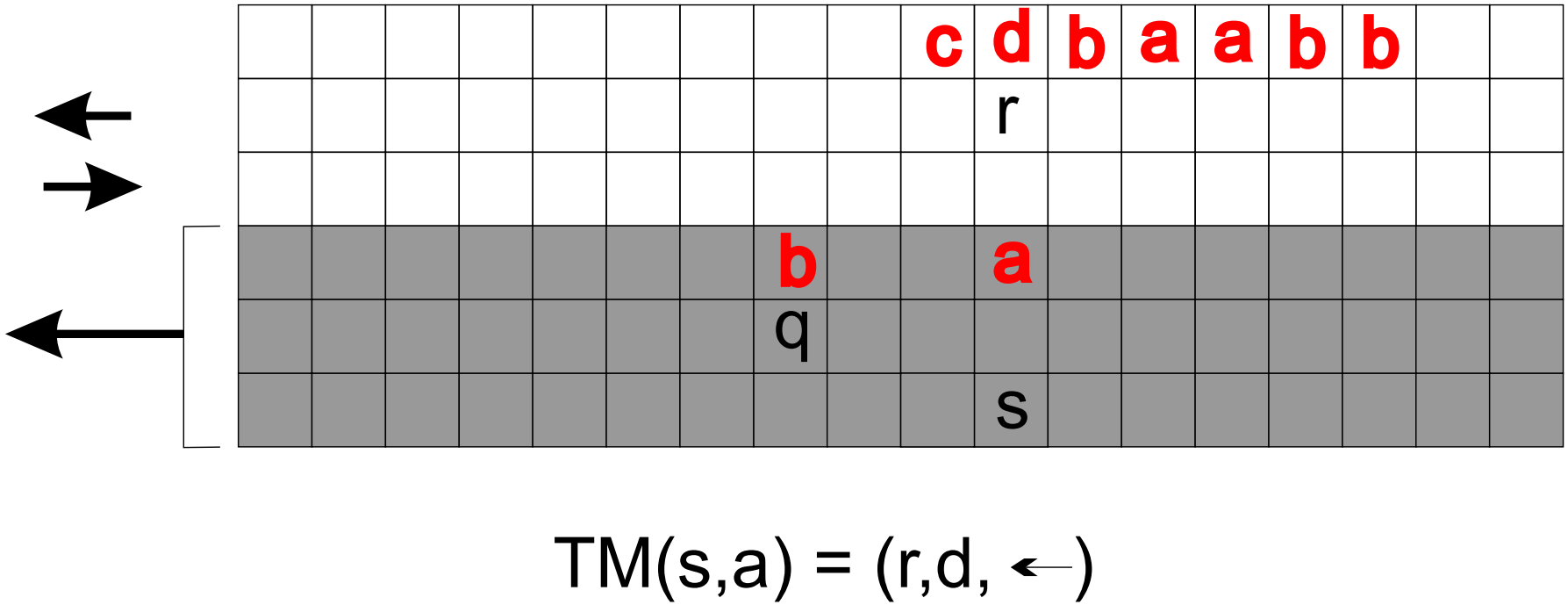

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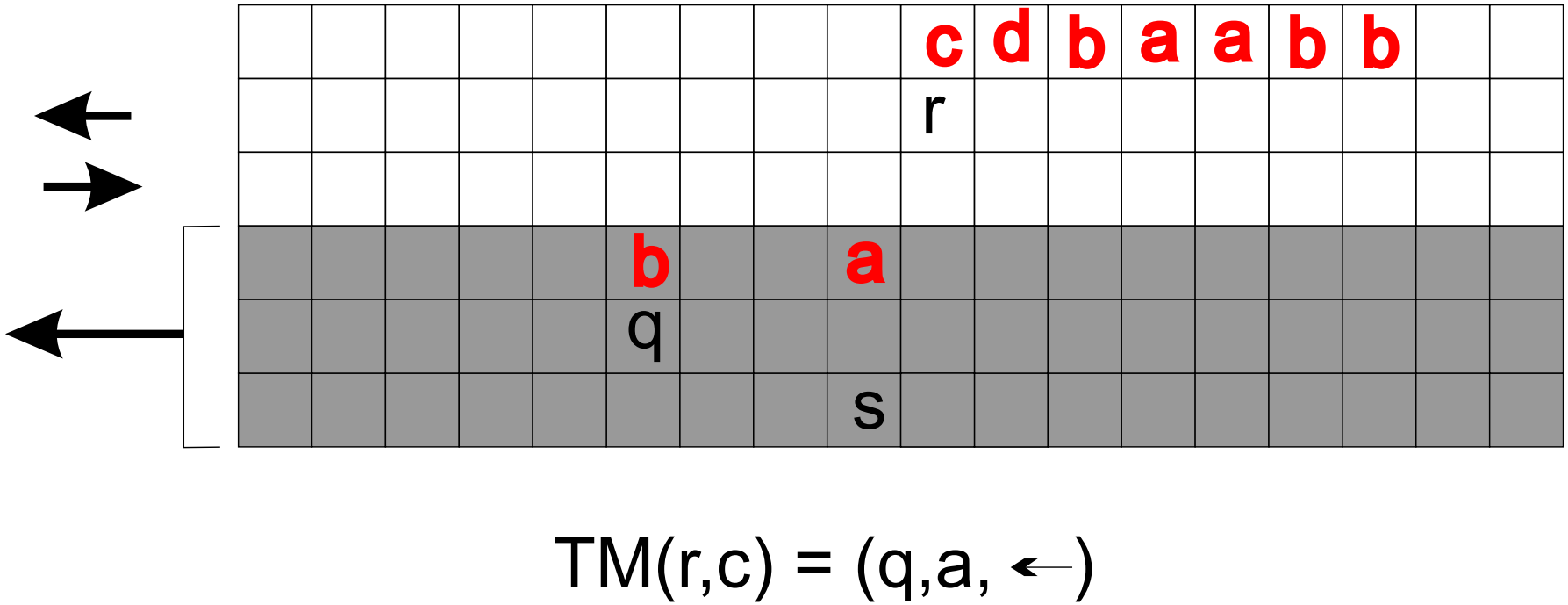

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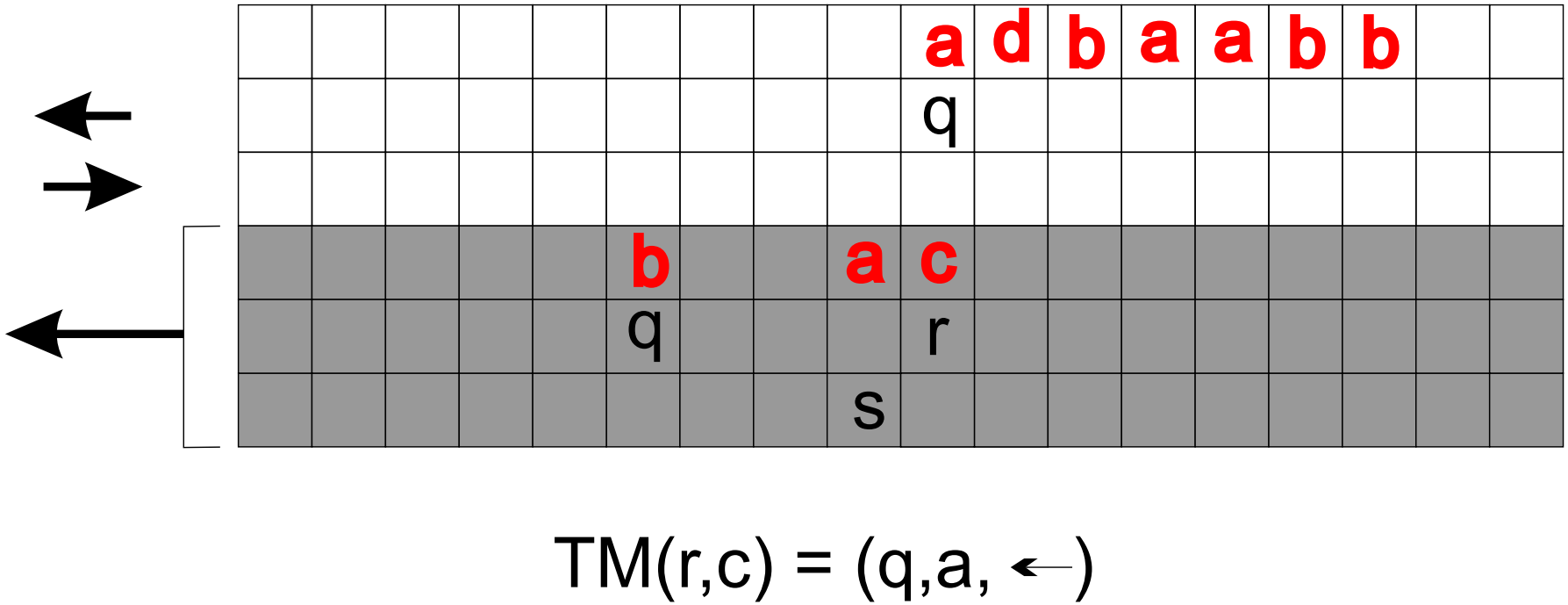

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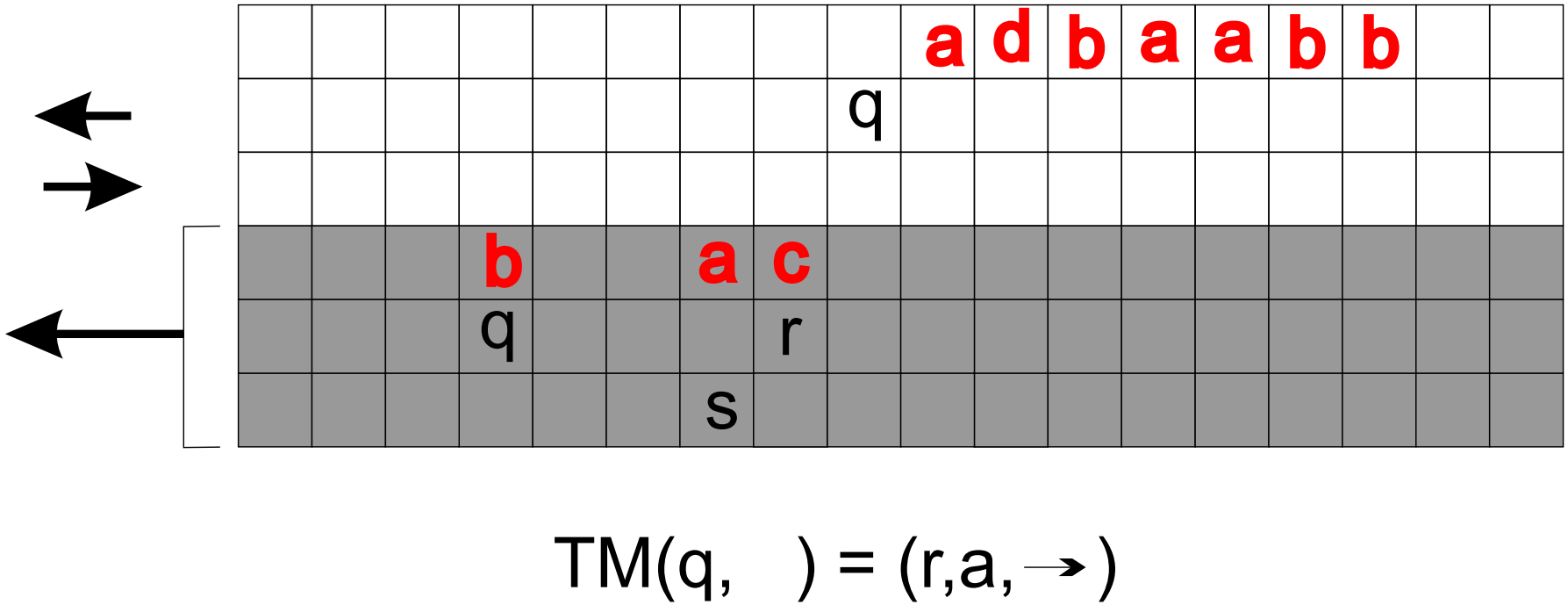

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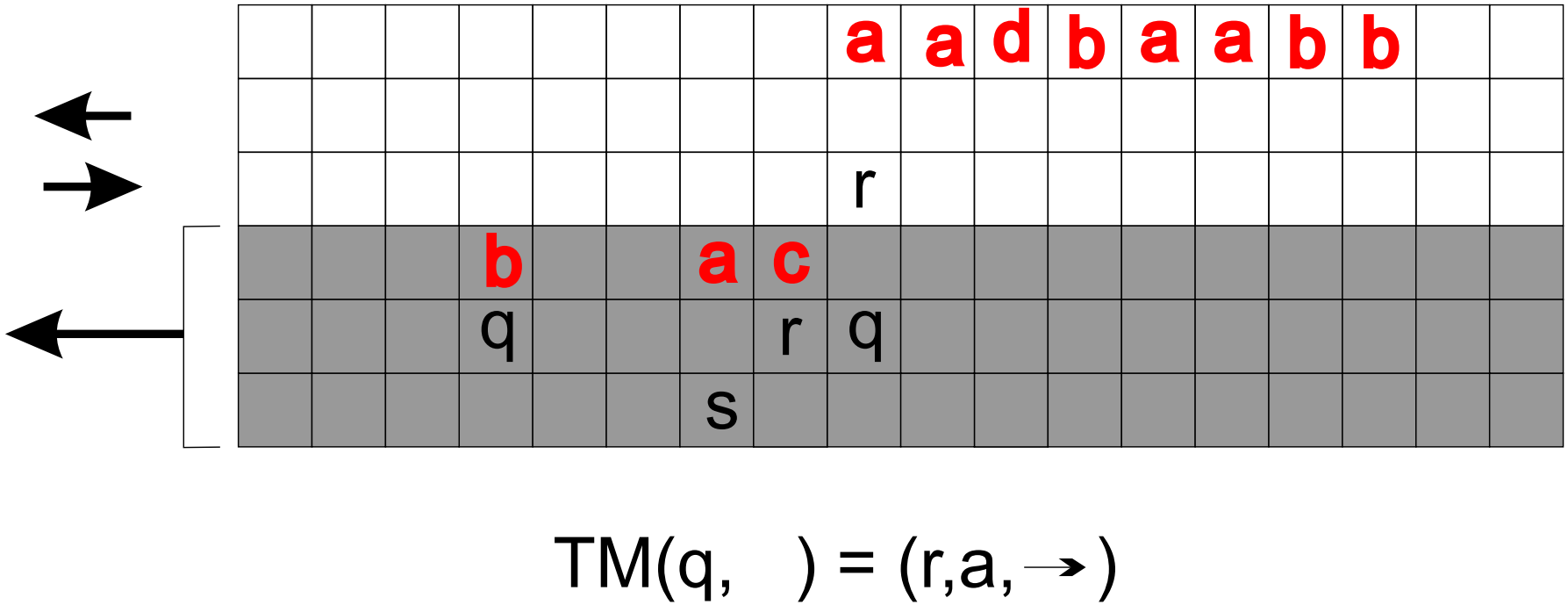

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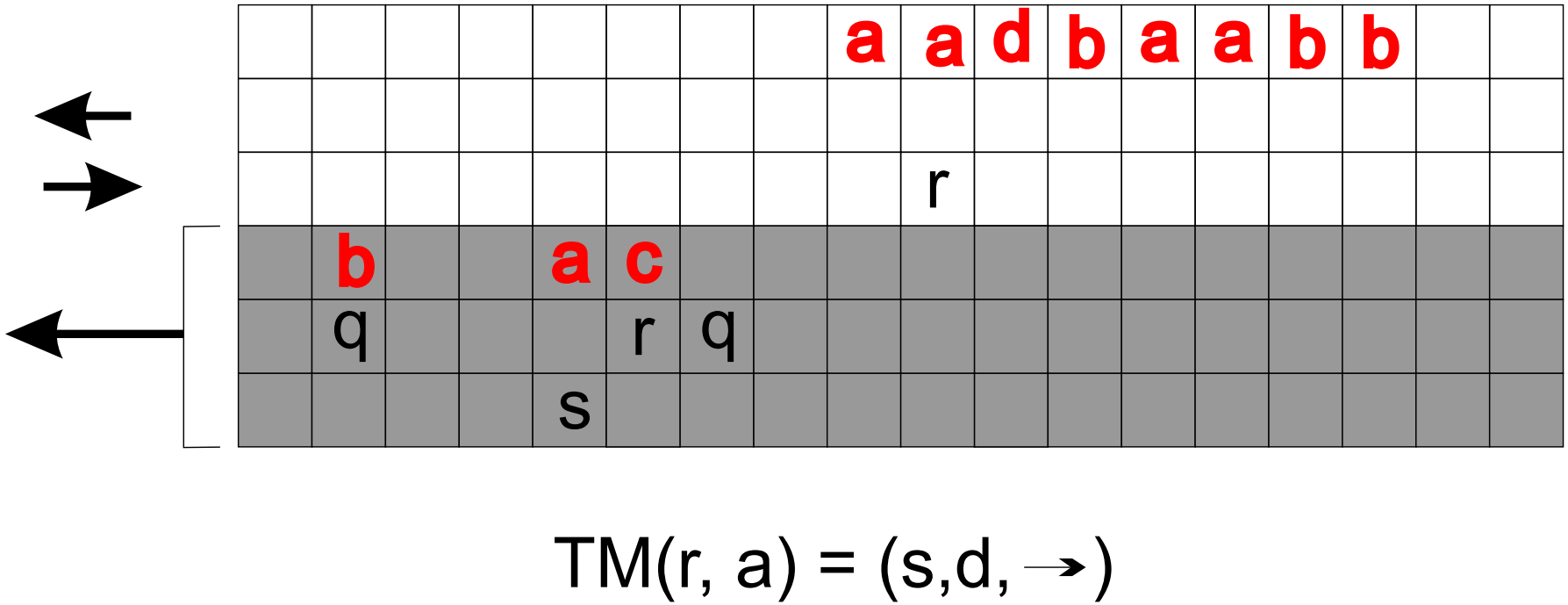
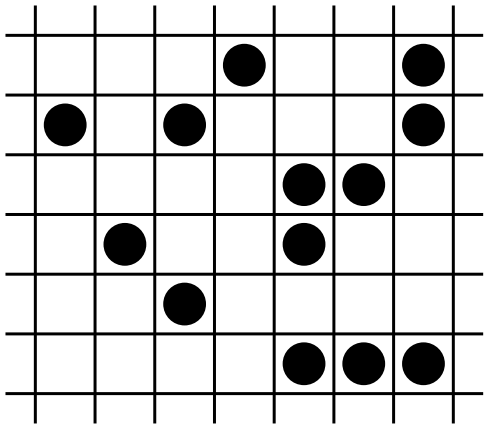


It is clear that the partially defined π is one-to-one. Any partially defined injective map $S \longrightarrow S$ can be completed into a bijection by matching the missing elements in the domain and range arbitrarily.

Note that the missing elements correspond to situations that never occur during valid simulations of the Turing machine (for example, when the incoming new "garbage bin" is not empty).

N.Margolus introduced a particularly simple two-dimensional RCA that is computationally universal, called the **Billiard Ball RCA**. This CA also introduced the technique of **block permutations** to guarantee reversibility.

In this technique the update is done in two steps:



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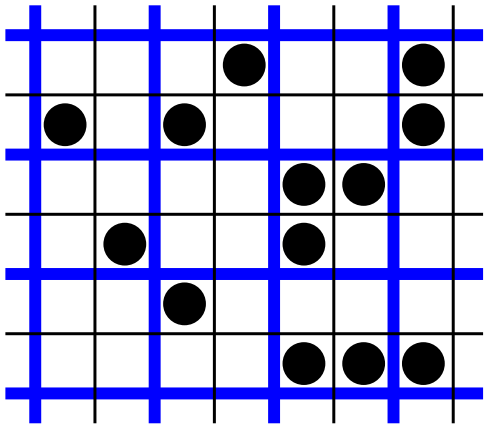

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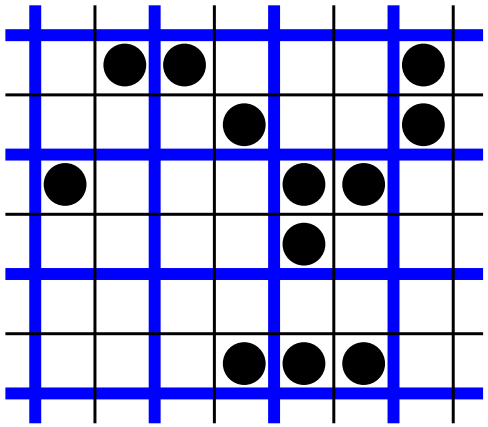

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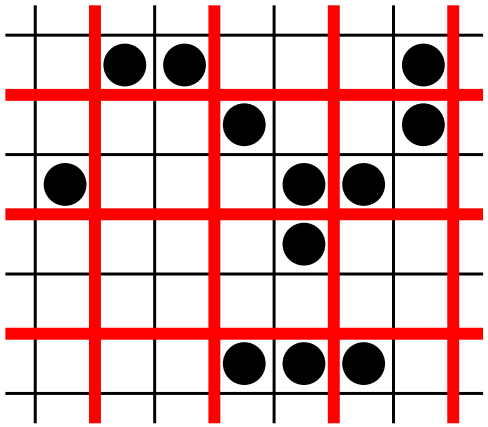

2. Shift the partitioning horizontally and vertically, and apply another permutation π_2 of S^4 on the new blocks.

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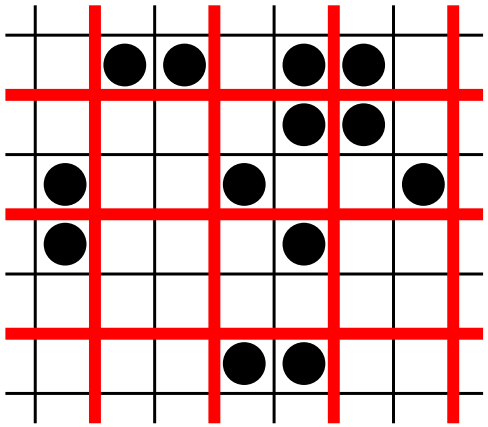

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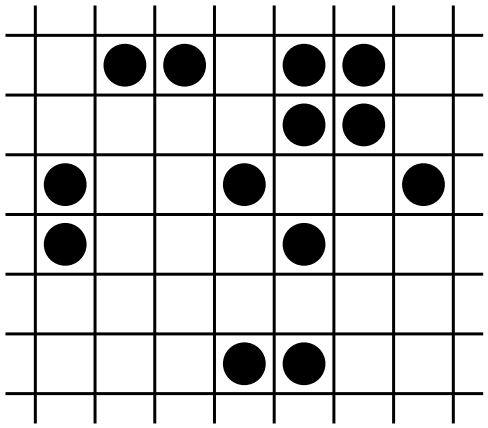
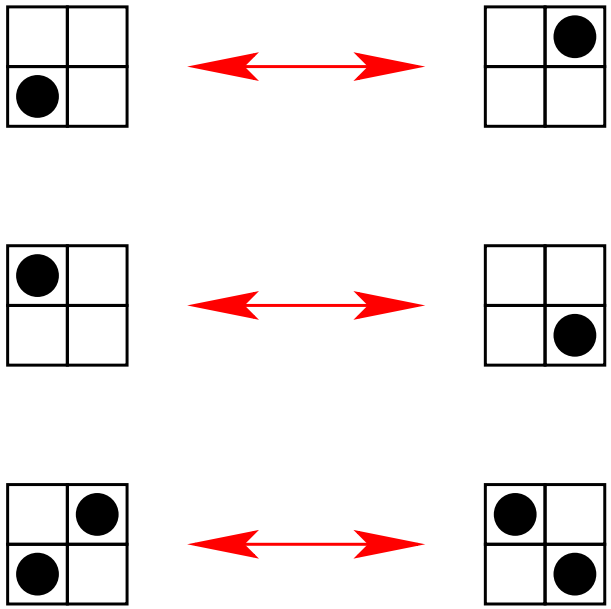
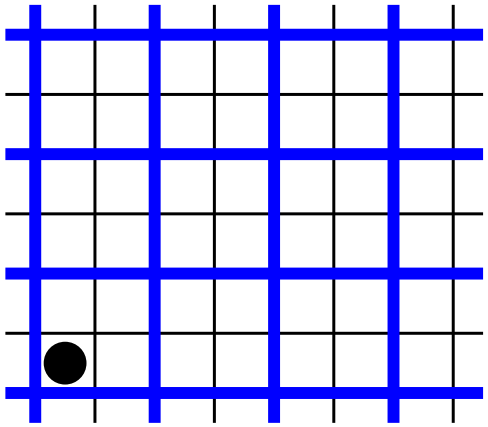


The composition of the two block permutation is one iteration of the CA. It is trivially reversible.

In the Billiard Ball RCA S is a binary set, and both permutations π_1 and π_2 are identical. They only make the following exchanges:



Because of alternating partitioning, a single black state propagates in one of the four diagonal directions:



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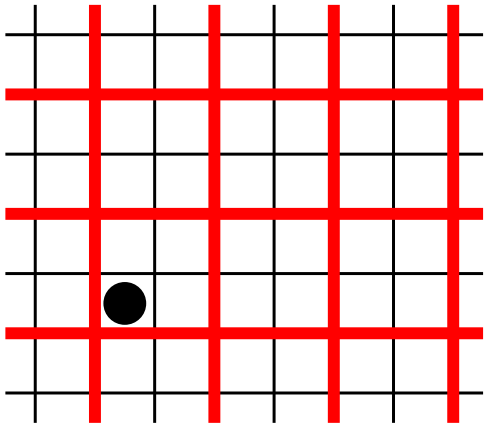

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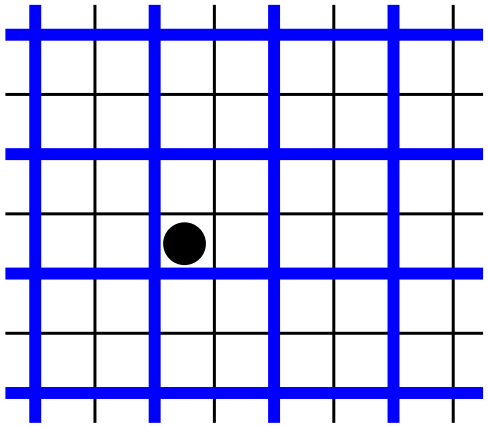

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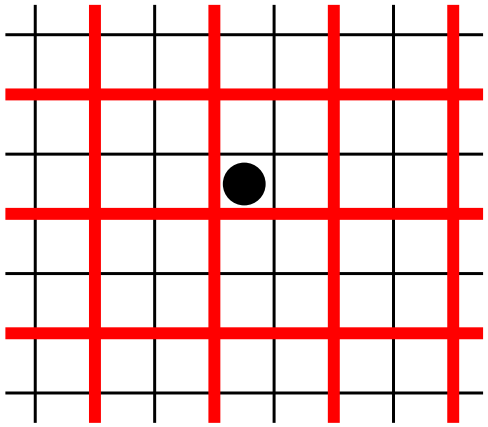

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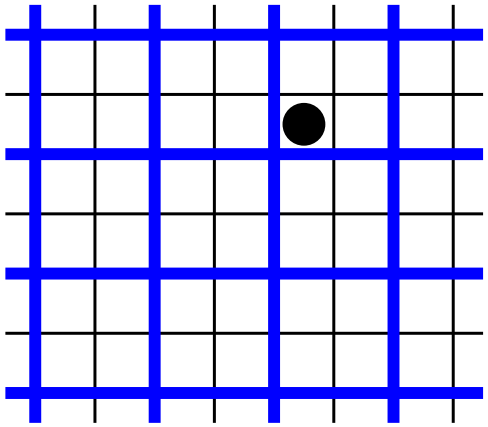

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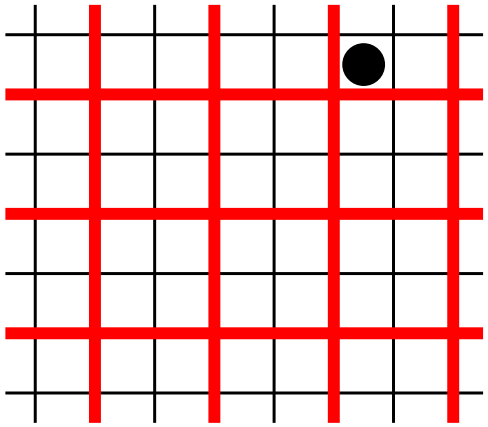
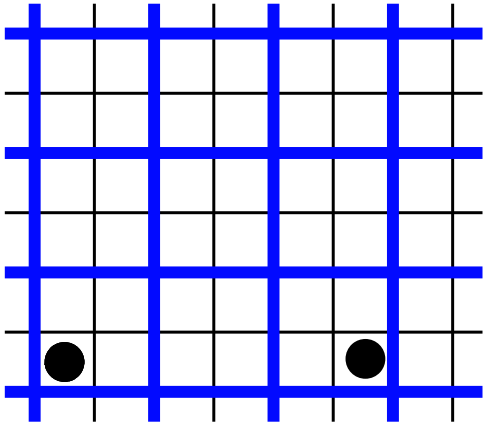


Two particles that collide sideways pause for a moment and reverse their direction:



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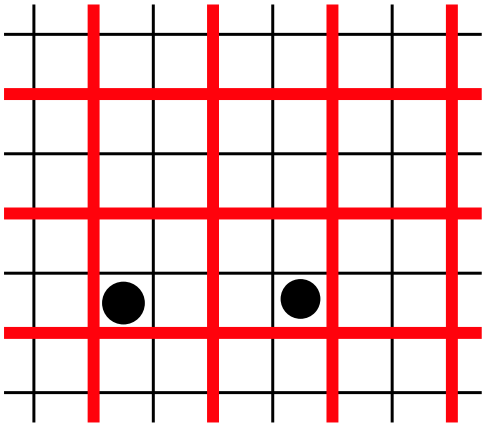

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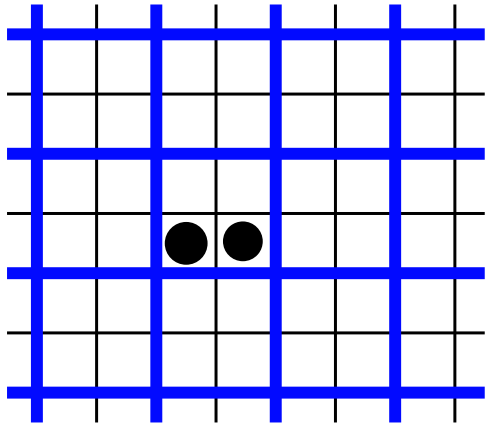

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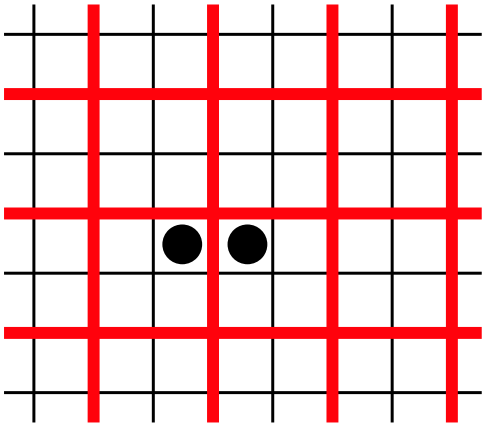

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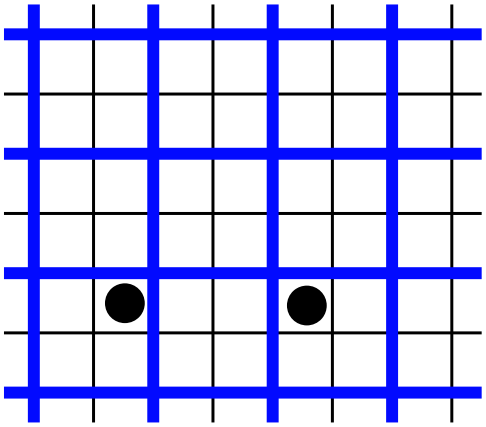

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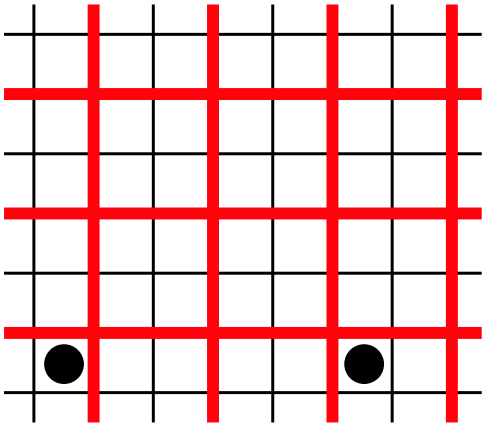

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Reversibility is trivially obtained using the block permutation technique. Also conservation laws are easy to enforce. For example, if the permutations π_1 and π_2 are such that they preserve the numbers of black states (e.g. as in the Billiard Ball RCA) then the whole CA conserves the number of black states.

Reversible CA come up naturally when simulating reversible physical systems.

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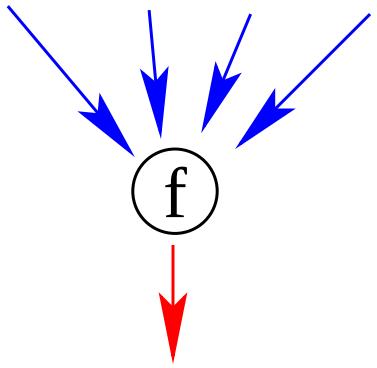
Conversely, physically building a cellular automata device in the reversible physical universe is most energy efficient if the CA is reversible. In fact, whenever a physical device erases information (by performing, say, a logical AND operation) it must dissipate energy from the system, usually in the form of heat.

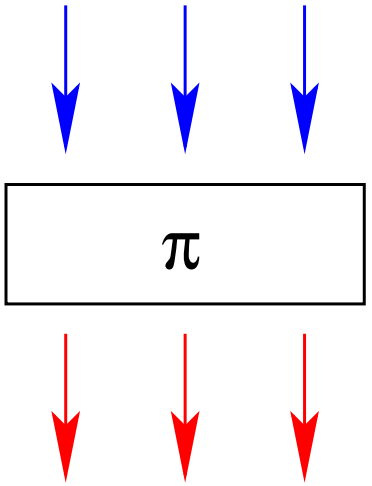
It is more energy efficient to use only reversible logical operations during computation.

The problem with reversible CA is that reversibility is **global**: the transformation $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$ is bijective on infinite configurations.

In order to exploit the reversibility of RCA they should be implemented using finite reversible logical gates. However, CA are given in terms of the local update rule $f : S^n \longrightarrow S$ that is **not** reversible:

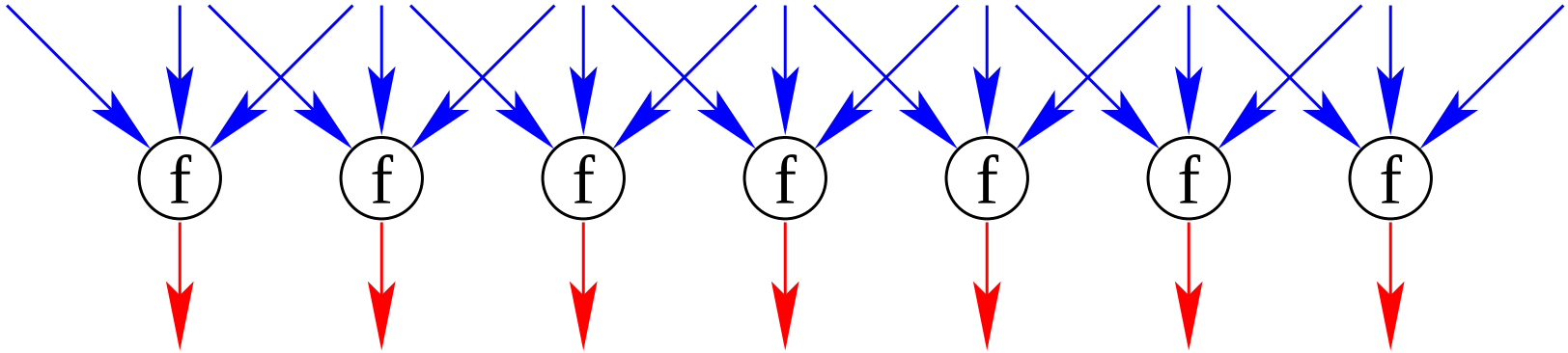


Rather than using the non-reversible local rule f we would need to use reversible local functions

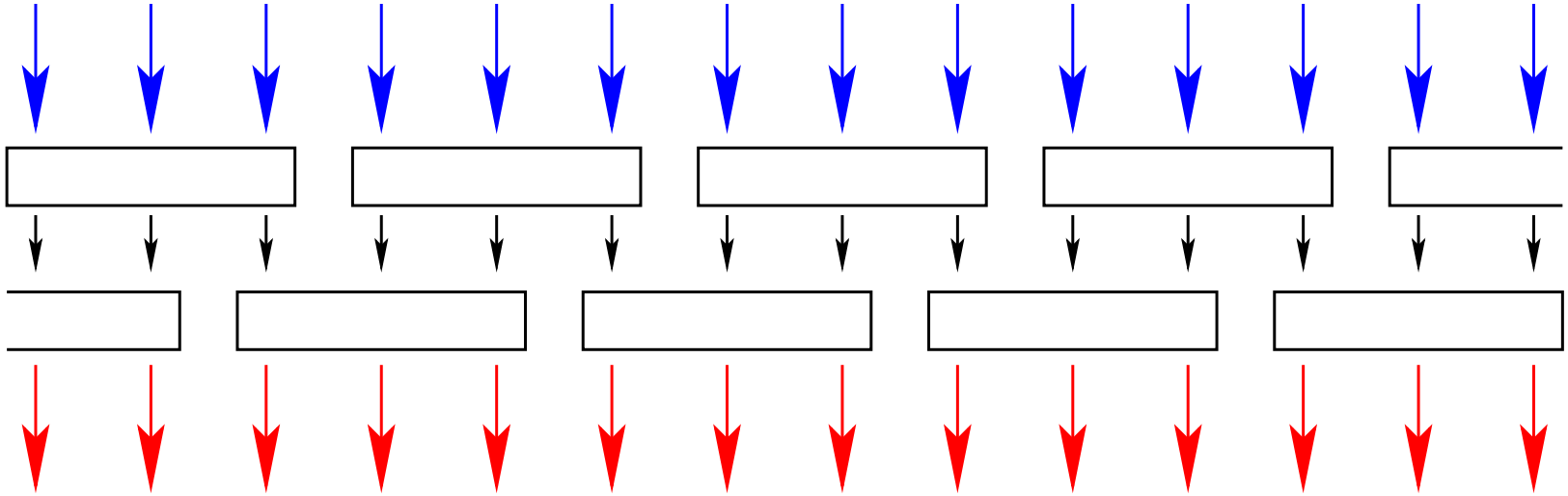


Partitioned CA, and the block partitioning technique of the Billiard Ball RCA are examples where the local rule was given in terms of reversible local rules. A natural question: Can every RCA be expressed using reversible local rules ?

In other words, we would like to replace a traditional CA



with something like



that computes the same global function $G : S^{\mathbb{Z}^d} \longrightarrow S^{\mathbb{Z}^d}$.

The answer is **positive** in one- and two-dimensional spaces:

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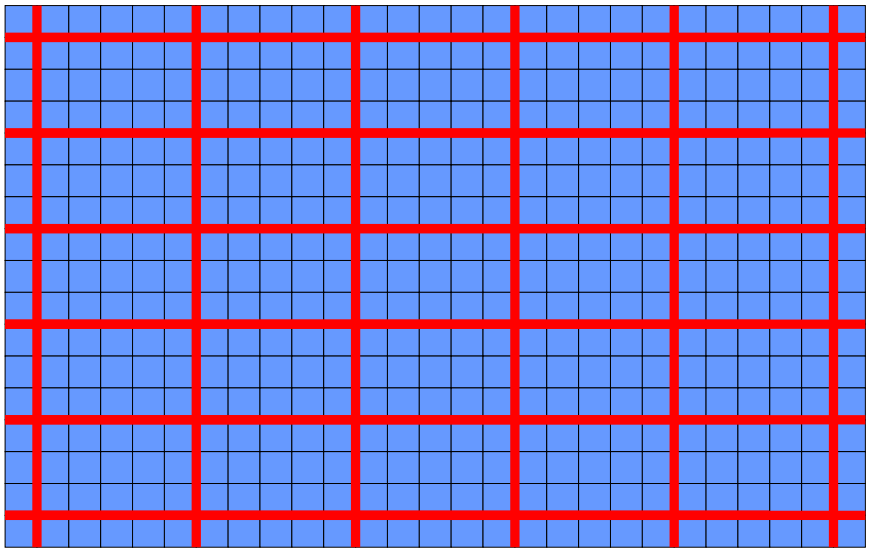
Terminology used: A **block permutation** consists of partitioning the space into congruent finite segments (rectangles, hypercubes in higher dimensions), and applying a permutation π inside every block:



The answer is **positive** in one- and two-dimensional spaces:

• Every one-dimensional RCA is a composition of two block permutations and a translation of a track.

Translating a track means expressing the state set as a cartesian product

$$S = S_1 \times S_2$$

and translating the S_1-components.

The answer is **positive** in one- and two-dimensional spaces:

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The question is **open** for three-dimensional CA.

Conjecture: Every d-dimensional RCA can be expressed as a composition of block permutations and a translation of a track.

(We know that if the conjecture holds then $d+1$ block permutations suffice.)

Outline of the talk(s)

(1) **Definitions** (configurations, neighborhoods, cellular automata, reversibility, injectivity, surjectivity)

(2) **Classical results** (Hedlund's theorem, balance in surjective CA, Garden-Of-Eden theorem)

(3) **Reversible CA** (universality, billiard ball CA)

(4) **Decidability questions** (Wang tiles, domino problem, snake tiles, undecidability of nilpotency, reversibility, surjectivity, periodicity)

(5) **Dynamical systems aspects** (equicontinuity classification, limit sets)

Decidability questions

Suppose we are given a CA (in terms of its local update rule) and want to know if it is reversible or surjective ? Is there an algorithm to decide this ? Or is there an algorithm to determine if the dynamics of a given CA is trivial in the sense that after a while all activity has died ?

It turns out that many such algorithmic problems are **undecidable**. In some cases there is an algorithm for one-dimensional CA while the two-dimensional case is undecidable.

A useful tool to obtain undecidability results is the concept of **Wang tiles** and the undecidable **tiling problem**.

A **Wang tile** is a unit square tile with colored edges. A tile set T is a finite collection of such tiles. A valid tiling is an assignment

$$\mathbb{Z}^2 \longrightarrow T$$

of tiles on infinite square lattice so that the abutting edges of adjacent tiles have the same color.

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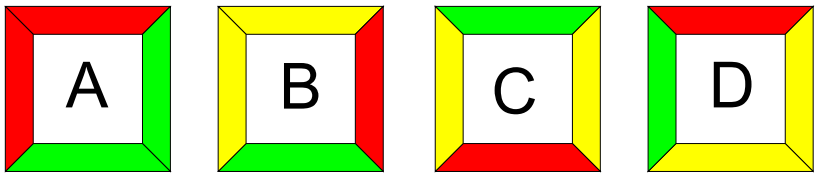
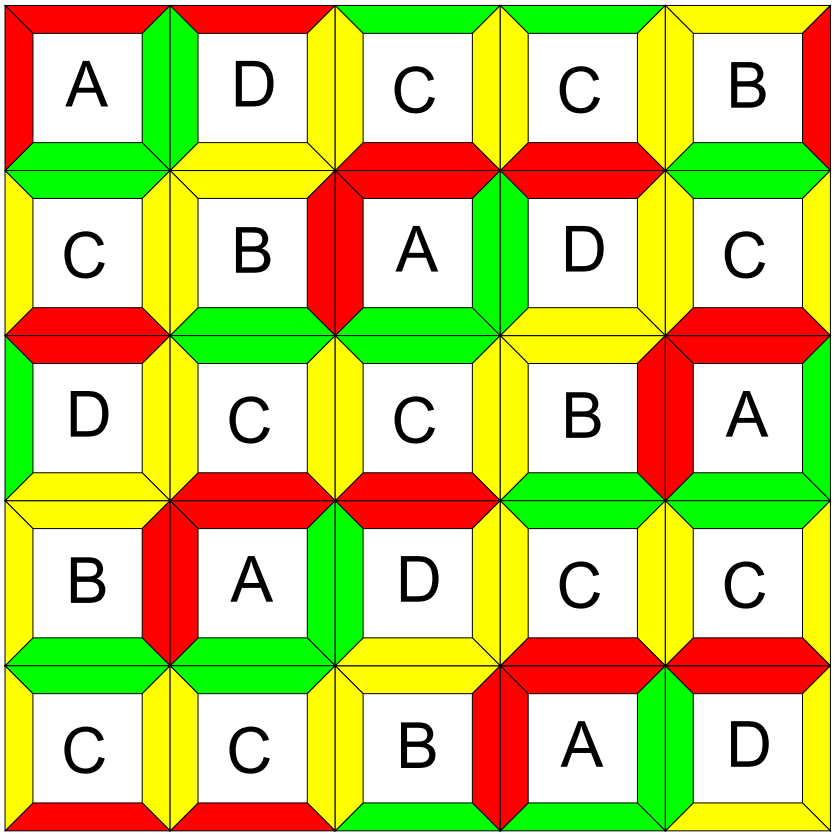
For example, consider Wang tiles



With copies of the given four tiles we can properly tile a 5×5 square...



... and since the colors on the borders match this square can be repeated to form a valid periodic tiling of the plane.

The set of valid tilings using elements of T is a translation invariant, compact subset of the configuration space $T^{\mathbb{Z}^2}$.

It is a two-dimensional counter-part of a **subshift of finite type** used in symbolic dynamics, as it can be defined via a finite collection of patterns that are not allowed in any valid tiling.

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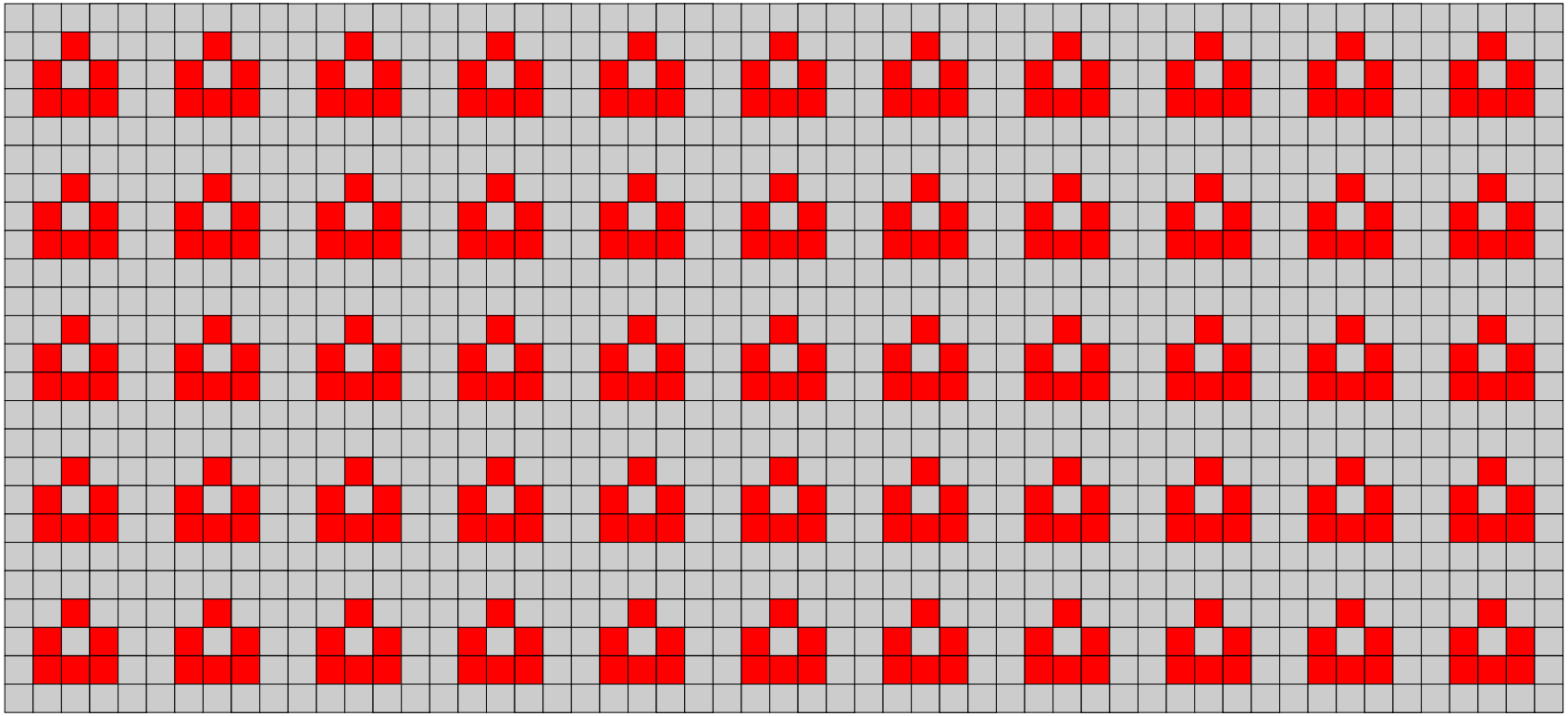
We also use the following terminology: Configuration $c \in T^{\mathbb{Z}^2}$ is **valid inside** $M \subseteq \mathbb{Z}^2$ if the colors match between any two neighboring cells, both of which are inside region M.

A configuration $c \in T^{\mathbb{Z}^2}$ is **periodic** if there are two linearly independent translations τ_1 and τ_2 that keep c invariant:

$$\tau_1(c) = \tau_2(c) = c.$$

Then c is also invariant under some horizontal and vertical translations.



More generally, a d-dimensional configuration $c \in S^{\mathbb{Z}^d}$ is **periodic** if it is invariant under d linearly independent translations.

For $d = 1$ this is just the usual periodicity of infinite words.

The **tiling problem** of Wang tiles is the decision problem to determine if a given finite set of Wang tiles admits a valid tiling of the plane.

Theorem (R.Berger 1966): The tiling problem of Wang tiles is undecidable.

Observations:

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Follows from compactness: Let C_n be the set of configurations that satisfy the tiling constraint inside the $(2n+1) \times (2n+1)$ square centered at the origin. Then

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \dots$$

is a decreasing chain of non-empty compact sets. Hence their intersection is non-empty.

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Follows from (1): Just try tiling larger and larger squares until (if ever) a square is found that can not be tiled.

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Reason: Just try tiling rectangles until (if ever) a valid tiling is found where colors on the top and the bottom match, and left and the right sides match as well.

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Follows from (2), (3) and undecidability of the tiling problem.

The tiling problem can be reduced to various decision problems concerning (two-dimensional) cellular automata, so that the undecidability of these problems then follows from Berger's result.

This is not so surprising since Wang tilings are "static" versions of "dynamic" cellular automata.

Example: Let us prove that it is undecidable whether a given two-dimensional CA G has any fixed point configurations, that is, configurations c such that $G(c) = c$.

Proof: Reduction from the tiling problem. For any given Wang tile set T (with at least two tiles) we effectively construct a two-dimensional CA with state set T, the von Neumann -neighborhood and a local update rule that keeps a tile unchanged if and only if its colors match with the neighboring tiles.

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Note: For one-dimensional CA it is decidable whether fixed points exist. Fixed points form a subshift of finite type that can be effectively constructed.

More interesting reduction: A CA is called **nilpotent** if all configurations eventually evolve into the quiescent configuration.

Observation: In a nilpotent CA all configurations must become quiescent within a bounded time, that is, there is number n such that $G^n(c)$ is quiescent, for all $c \in S^{\mathbb{Z}^d}$.

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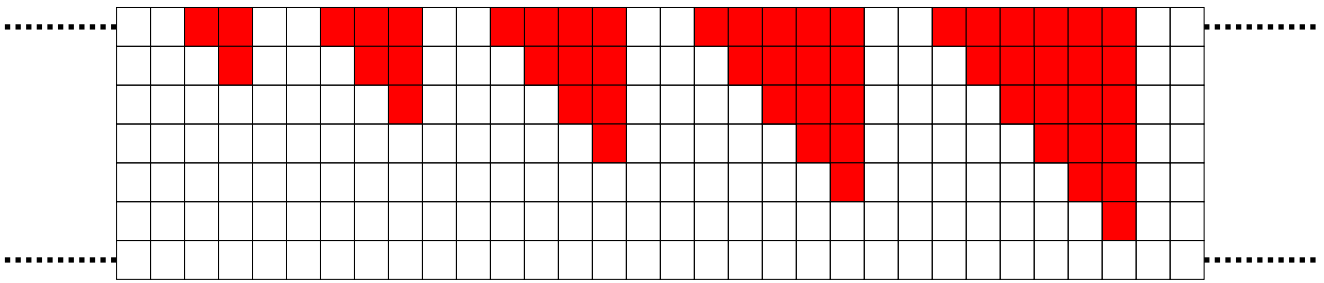
Observation: In a nilpotent CA all configurations must become quiescent within a bounded time, that is, there is number n such that $G^n(c)$ is quiescent, for all $c \in S^{\mathbb{Z}^d}$.

Proof: Suppose contrary: for every n there is a configuration c_n such that $G^n(c_n)$ is not quiescent. Then c_n contains a finite pattern p_n that evolves in n steps into some non-quiescent state. A configuration c that contains a copy of every p_n never becomes quiescent, contradicting nilpotency.



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Proof: For any given set T of Wang tiles the goal is to construct a two-dimensional CA that is nilpotent if and only if T does not admit a tiling.

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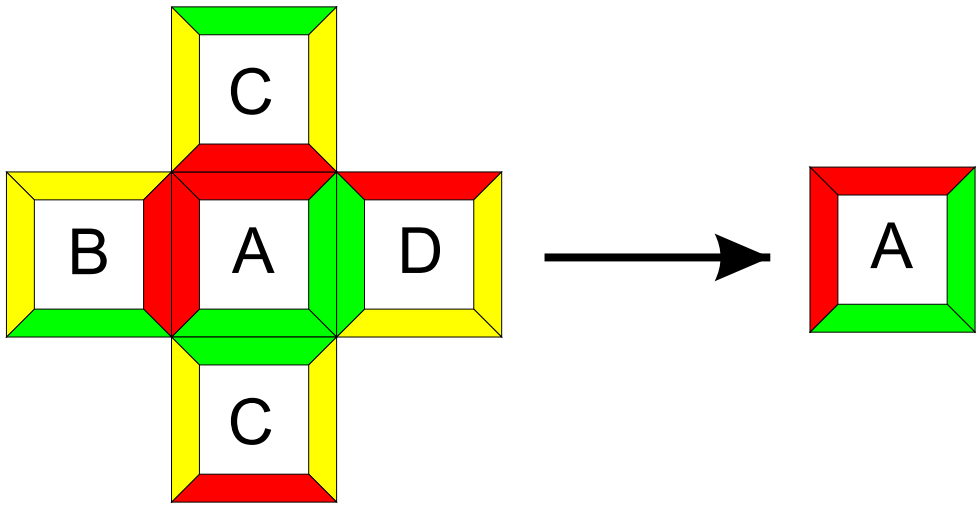

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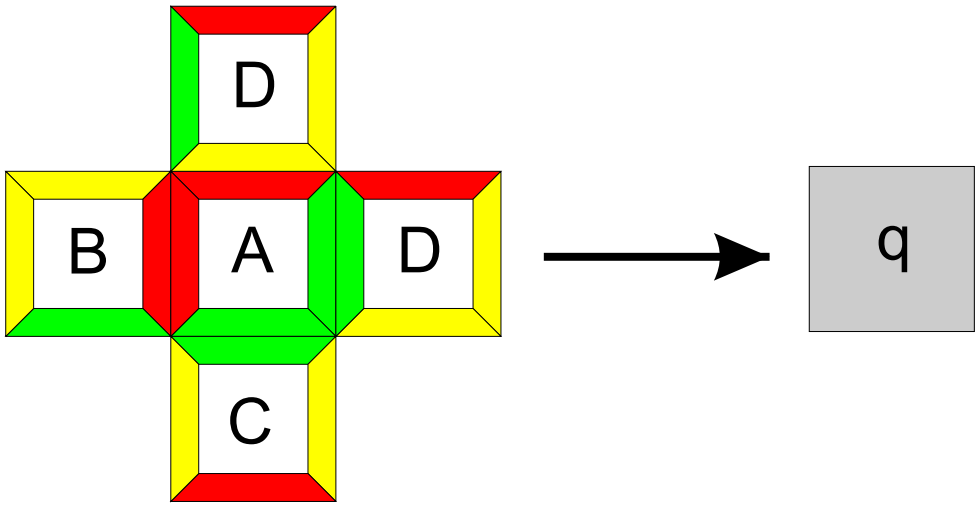

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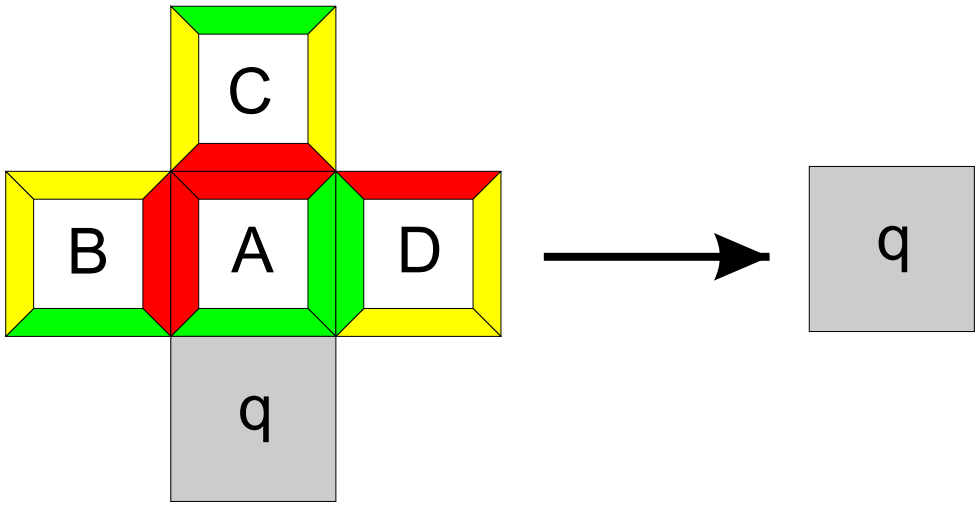

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- The local rule keeps state unchanged if all states in the neighborhood are tiles and the tiling constraint is satisfied. In all other cases the new state is q.

$\Longrightarrow$ If T admits a tiling c then c is a non-quiescent fixed point of the CA. So the CA is not nilpotent.

$\Longleftarrow$ If T does not admit a valid tiling then every $n \times n$ square contains a tiling error, for some n. State q propagates, so in at most $2n$ steps all cells are in state q. The CA is nilpotent. $\square$

If we do the previous construction for an aperiodic tile set T we obtain a two-dimensional CA in which every periodic configuration becomes eventually quiescent, but there are some non-periodic fixed points.

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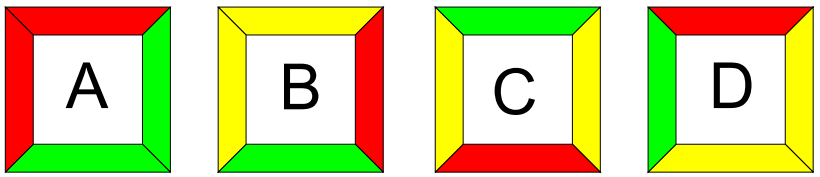
Another interesting observation is that while in nilpotent CA all configurations become quiescent within bounded time, that transient time can be very long: one cannot compute any upper bound on it as otherwise nilpotency could be effectively checked.

While tilings relate naturally to two-dimensional CA, one can still strengthen Berger's result so that the nilpotency can be proved undecidable for one-dimensional CA as well.

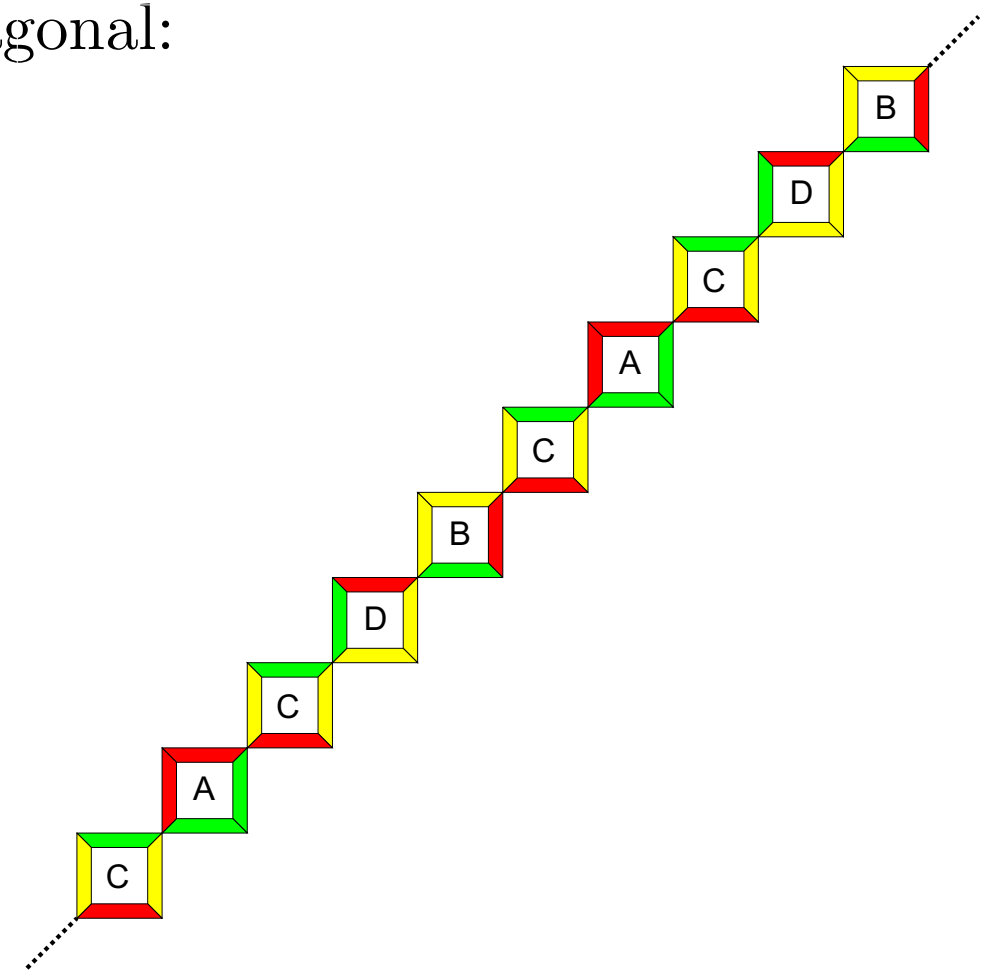
Tile set T is **NW-deterministic** if no two tiles have identical colors on their top edges and on their left edges. In a valid tiling the left and the top neighbor of a tile uniquely determine the tile.

For example, our sample tile set



is NW-deterministic.

In any valid tiling by NW-deterministic tiles, NE-to-SW diagonals uniquely determine the next diagonal below them. The tiles of the next diagonal are determined locally from the previous diagonal:



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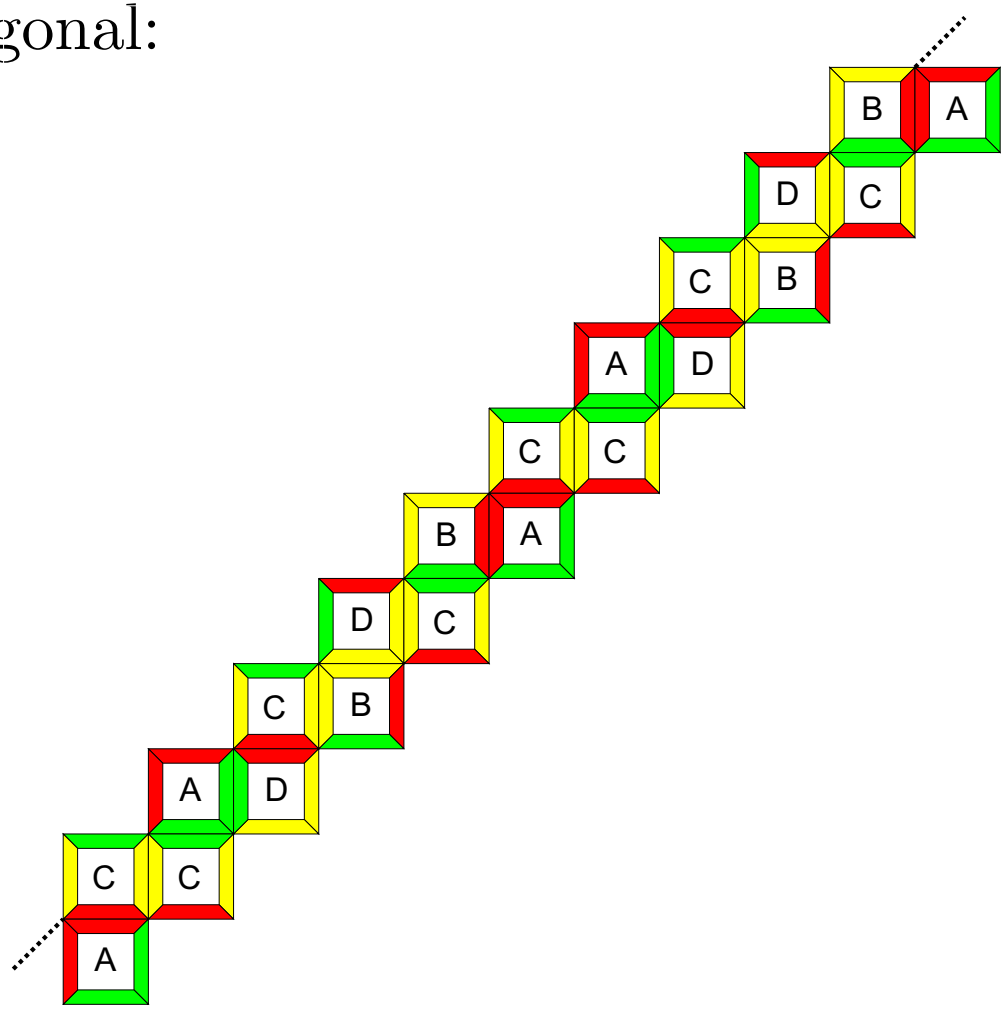

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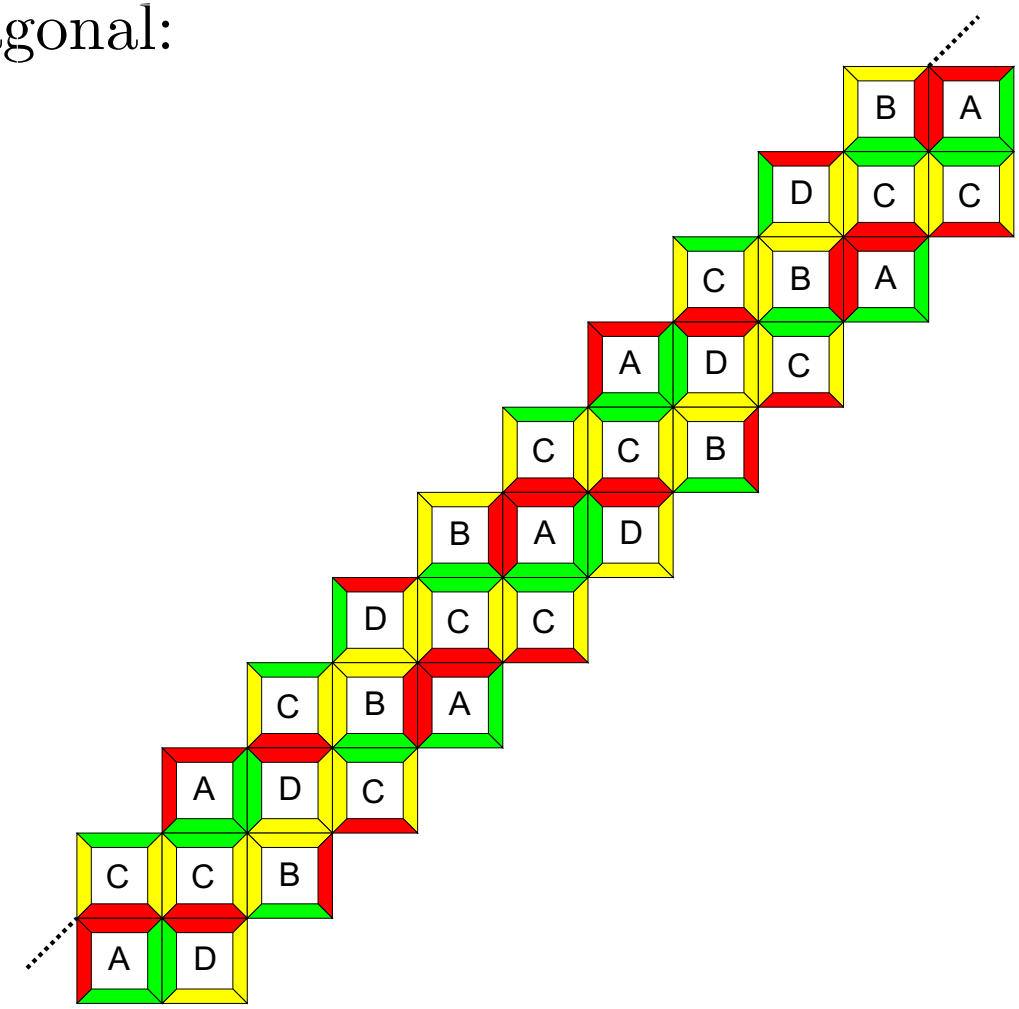

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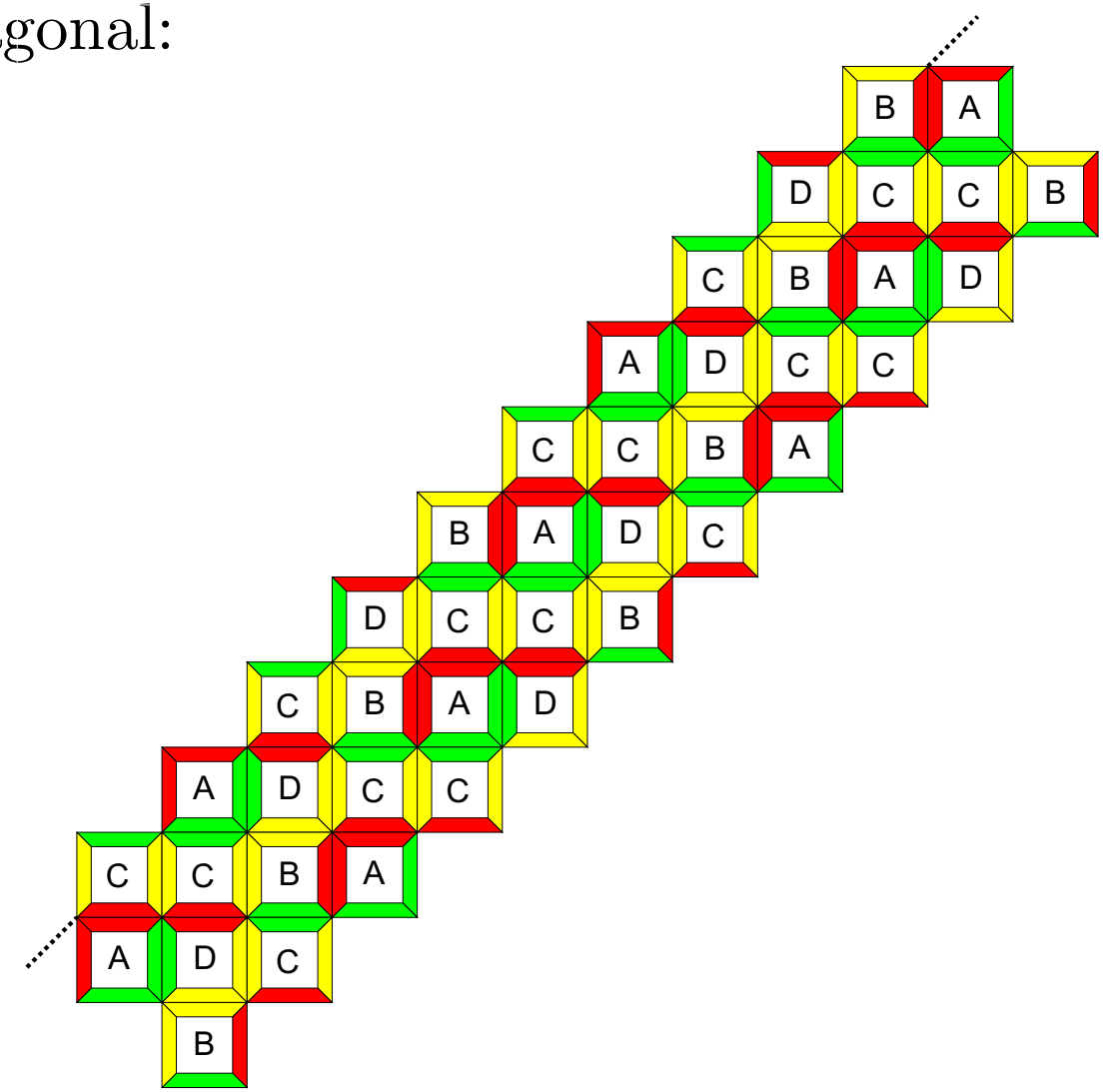

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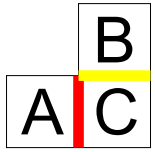
- state set is $S = T \cup \{q\}$ where q is a new symbol $q \notin T$,
- neighborhood is $(0, 1)$,
- local rule $f : S^2 \longrightarrow S$ is defined as follows:
  - $f(A, B) = C$ if the colors match in 

  - $f(A, B) = q$ if $A = q$ or $B = q$ or no matching tile C exists.

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$\Longleftarrow$ If T does not admit a valid tiling then every $n \times n$ square contains a tiling error, for some n. Hence state q is created inside every segment of length n. Since q starts spreading once it has been created, the whole configuration becomes eventually quiescent.

Now we just need the following strengthening of Berger's theorem:

Theorem: The tiling problem is undecidable among NW-deterministic tile sets.

and we have

Theorem: It is undecidable whether a given one-dimensional CA (with spreading state q) is nilpotent. □

If we do our construction using an aperiodic NW-deterministic tile set (these exist!) then we have an interesting one-dimensional CA in which all periodic configurations eventually die, but there are non-periodic configurations that never create a quiescent state in any cell.

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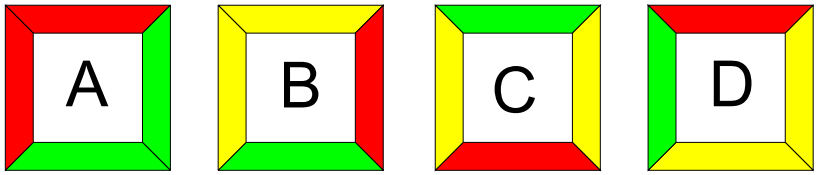
As in the two-dimensional case, the transient time before a one-dimensional nilpotent CA dies can be very long: it cannot be bounded by any computable function.

Analogously we can define NE-, SW- and SE-determinism of tile sets. A tile set is called **4-way** deterministic if it is deterministic in all four corners.

Our sample tile set is 4-way deterministic



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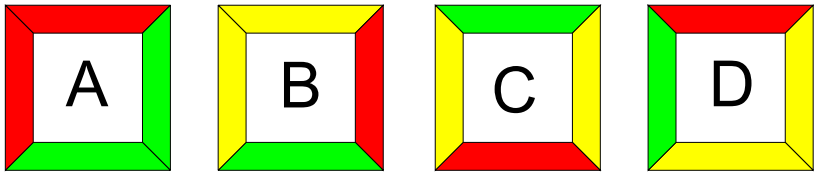


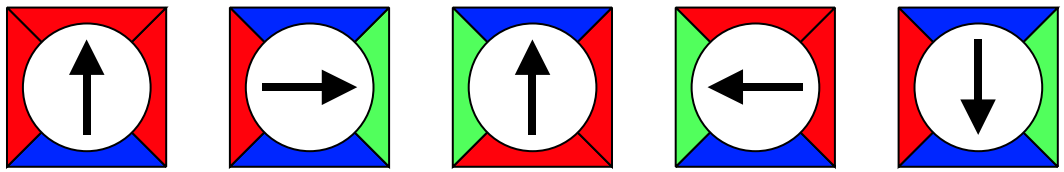
We recently showed the following:

Theorem: The tiling problem is undecidable among 4-way deterministic tile sets.

This result provides some undecidability results for dynamics of reversible one-dimensional CA.

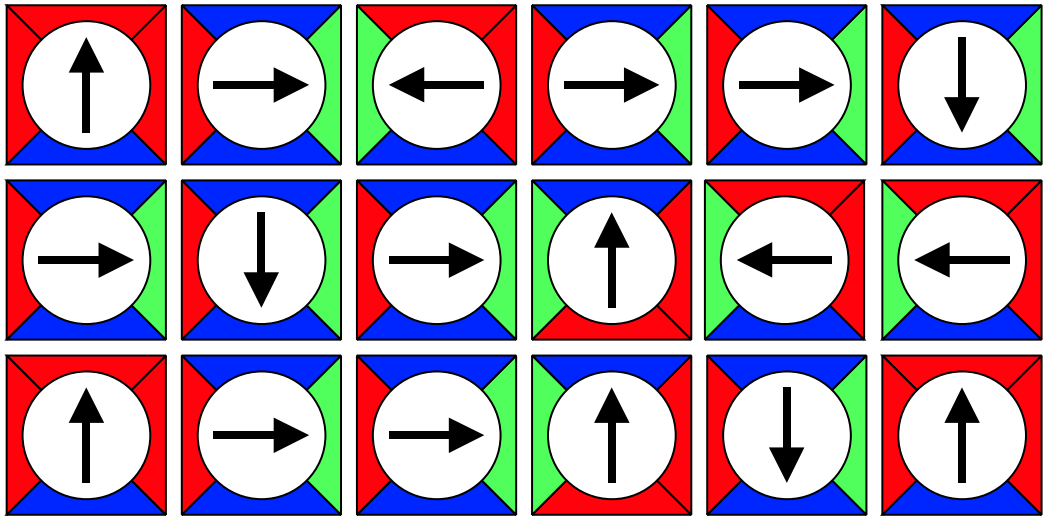
Next: Some interesting (and useful) properties of a particular tile set called SNAKES.

In addition to colored edges, these tiles also have an arrow printed on them. The arrow is horizontal or vertical and it points to one of the four neighbors of the tile:



Such tiles with arrows are called **directed tiles**.

Given a configuration (valid tiling or not!) and a starting position, the arrows specify a path on the plane. Each position is followed by the neighboring position indicated by the arrow of the tile:



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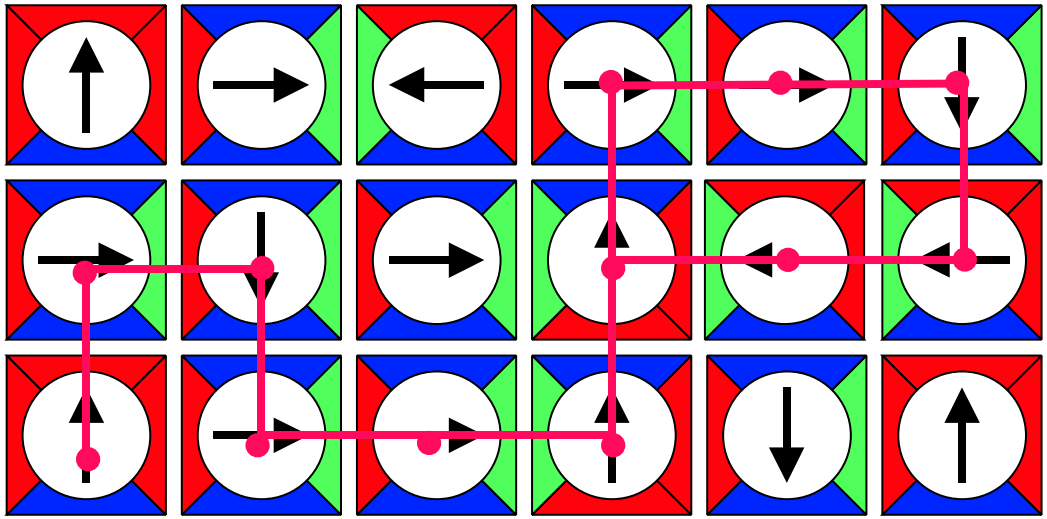

The path may enter a loop...

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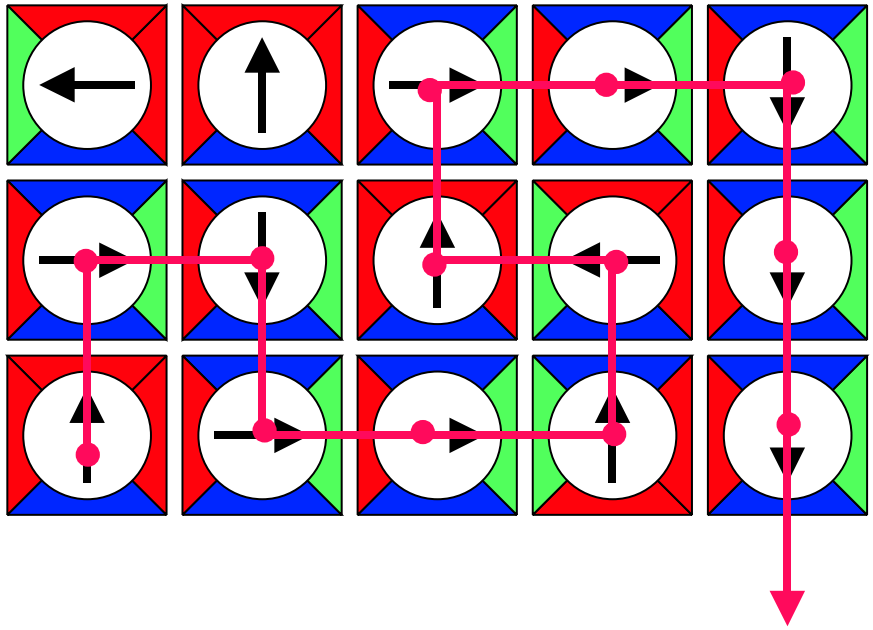
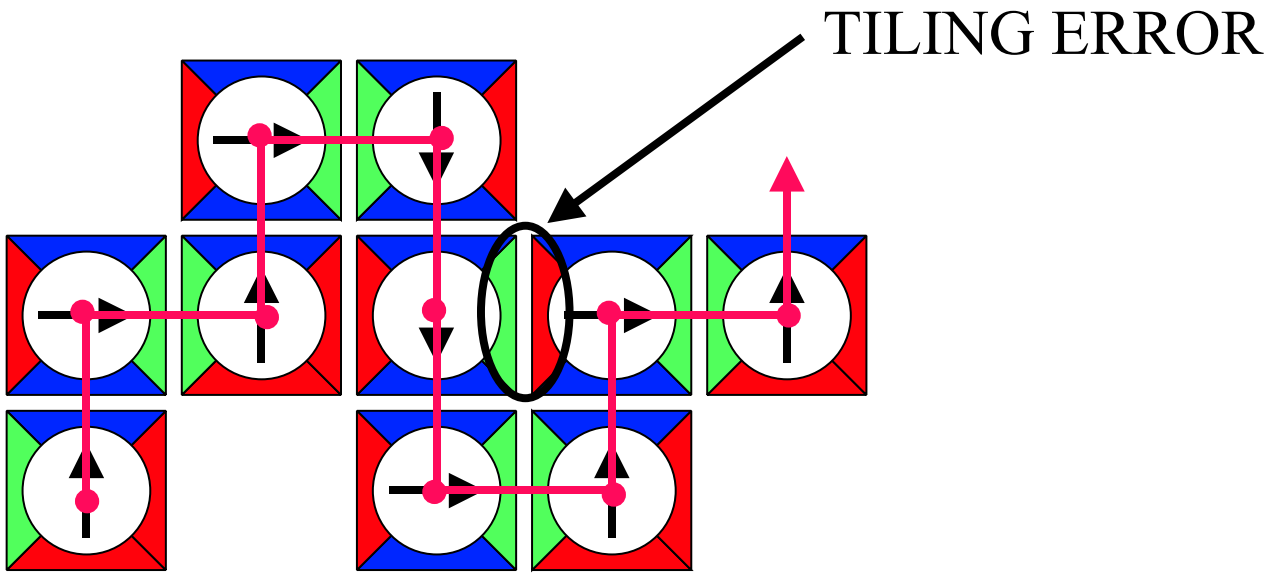


. . . or the path may be infinite and never return to a tile visited before.

The directed tile set SNAKES has the following property: On any configuration (valid tiling or not) and on any path that follows the arrows one of the following two things happens:

(1) Either there is a tiling error between two tiles both of which are on the path,



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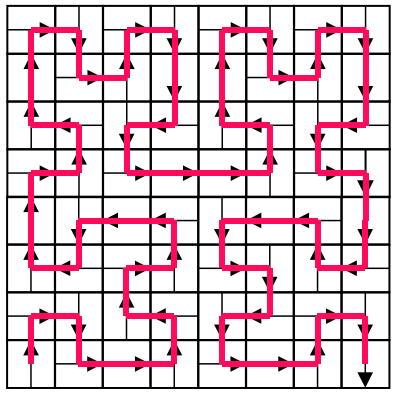
(2) or the path is a plane-filling path, that is, for every positive integer n there exists an $n \times n$ square all of whose positions are visited by the path.



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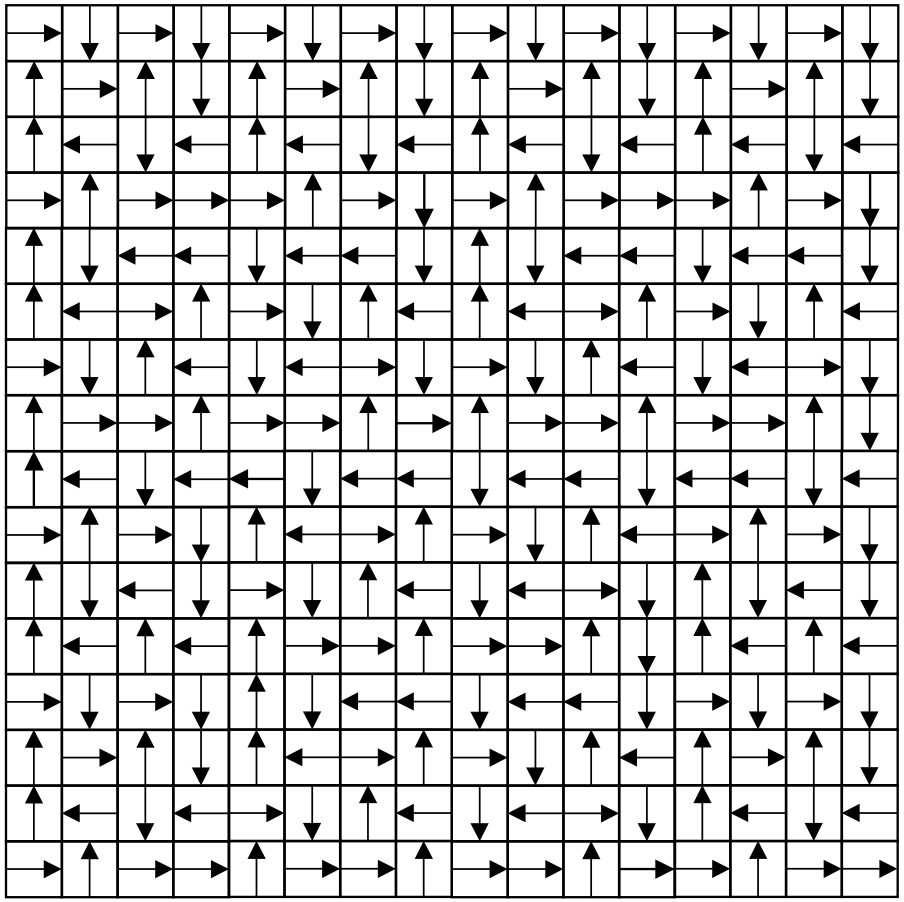
Note that the tiling may be invalid outside path P, yet the path is forced to snake through larger and larger squares.

SNAKES also has the property that it admits a valid tiling.

The paths that SNAKES forces when no tiling error is encountered have the shape of the well known plane-filling Hilbert-curve



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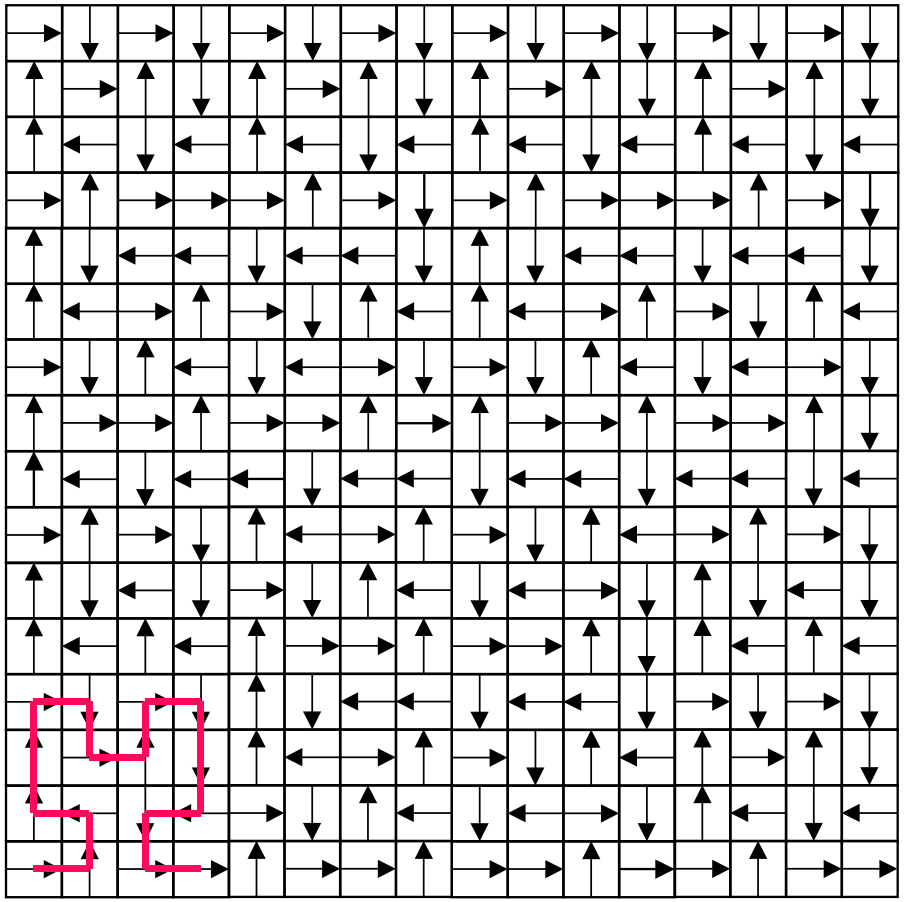

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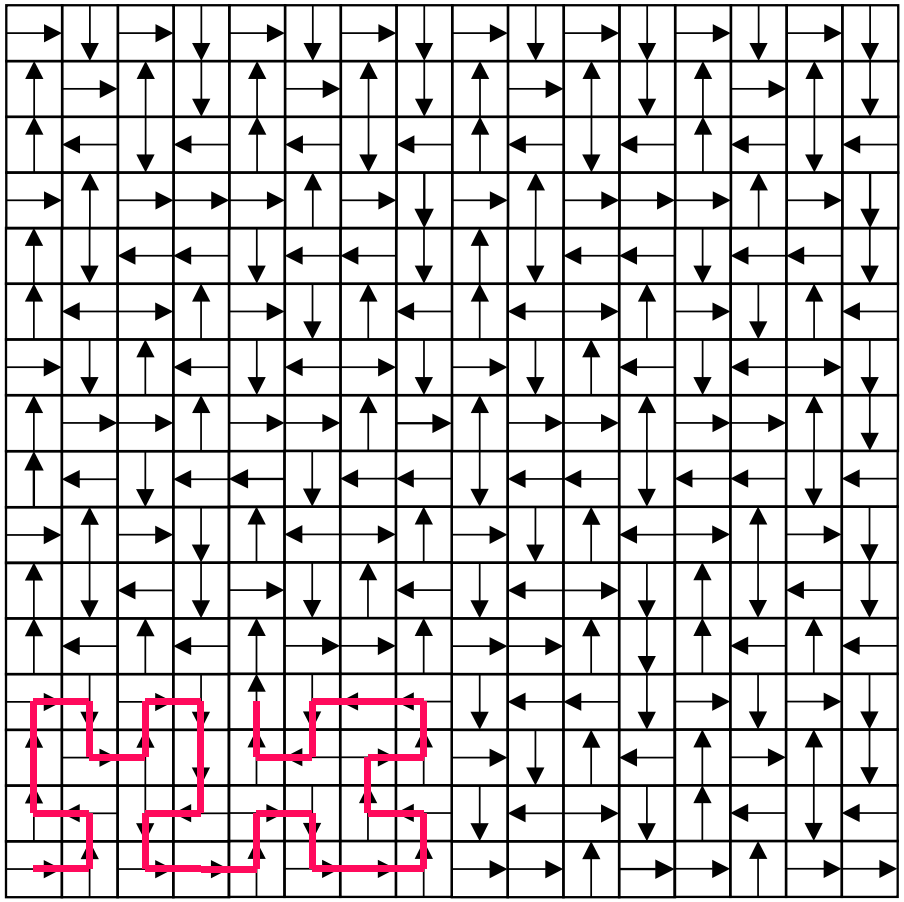

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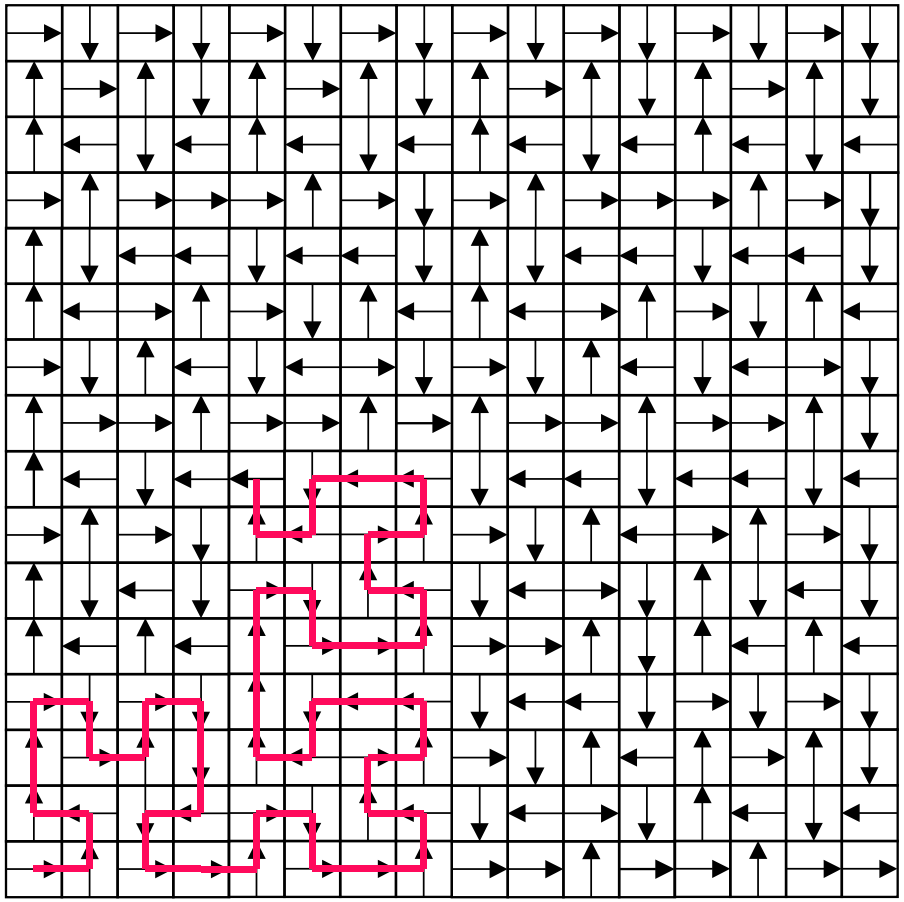

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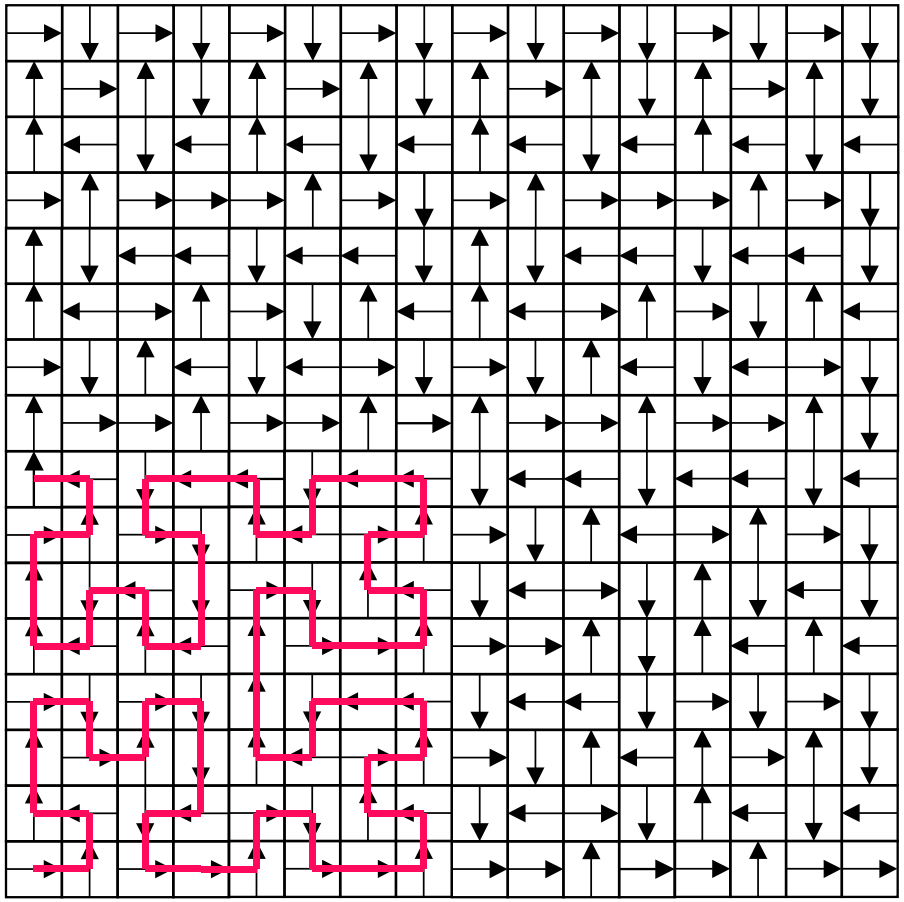

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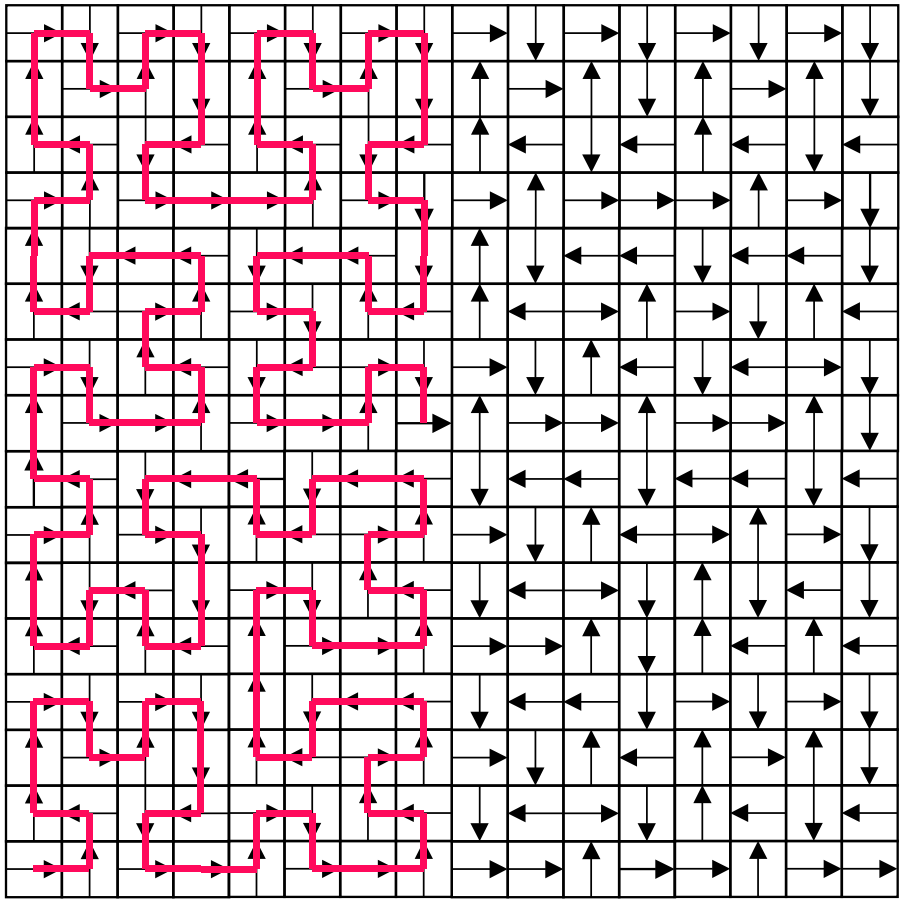

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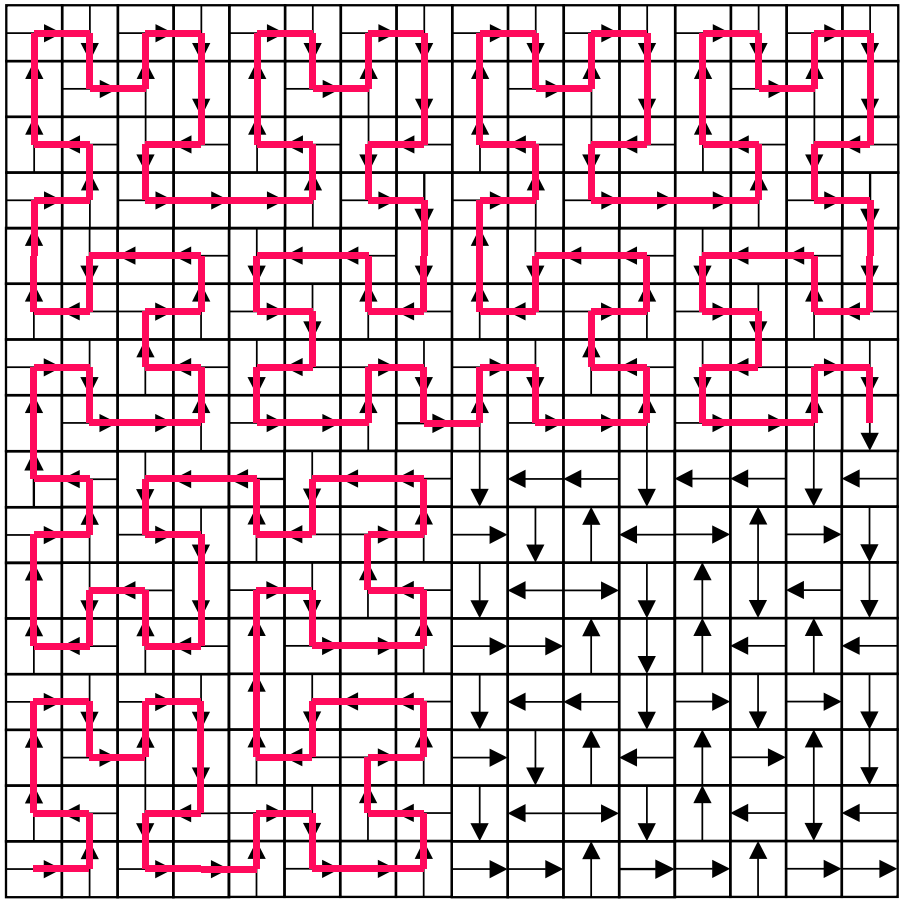

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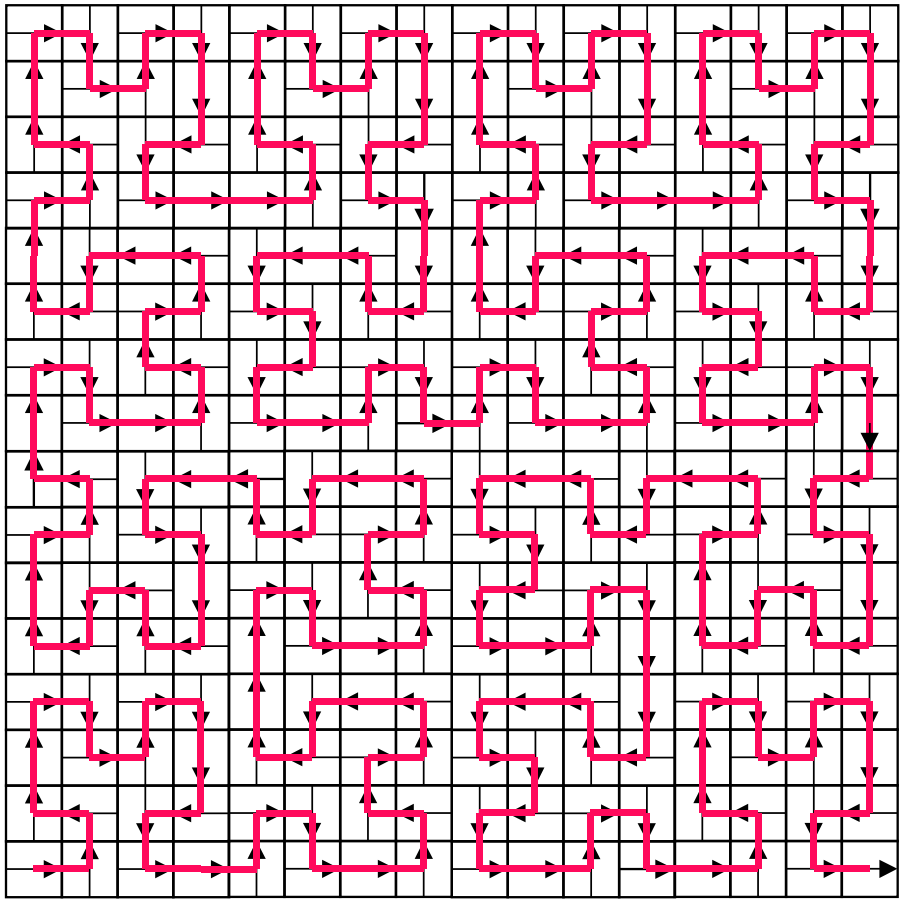

First application of SNAKES: An example of a two-dimensional CA that is injective on periodic configurations but is not injective on all configurations.

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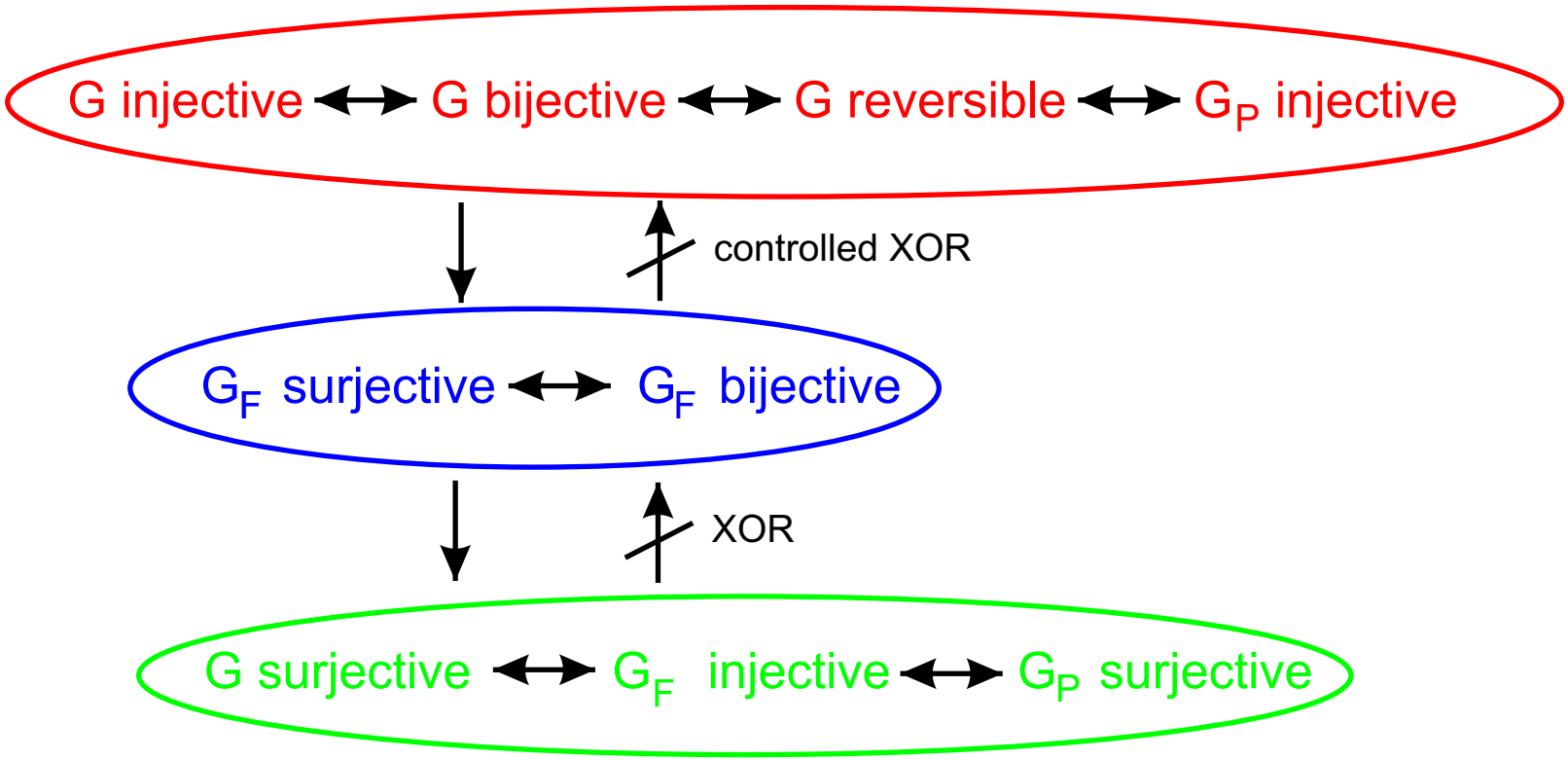
Let G_P denote the restriction of CA function G into periodic configurations.

Among one-dimensional CA the following facts hold:

$$G \text{ injective} \iff G_P \text{ injective},$$

$$G \text{ surjective} \iff G_P \text{ surjective}.$$



Among two-dimensional CA only the implications are easy:

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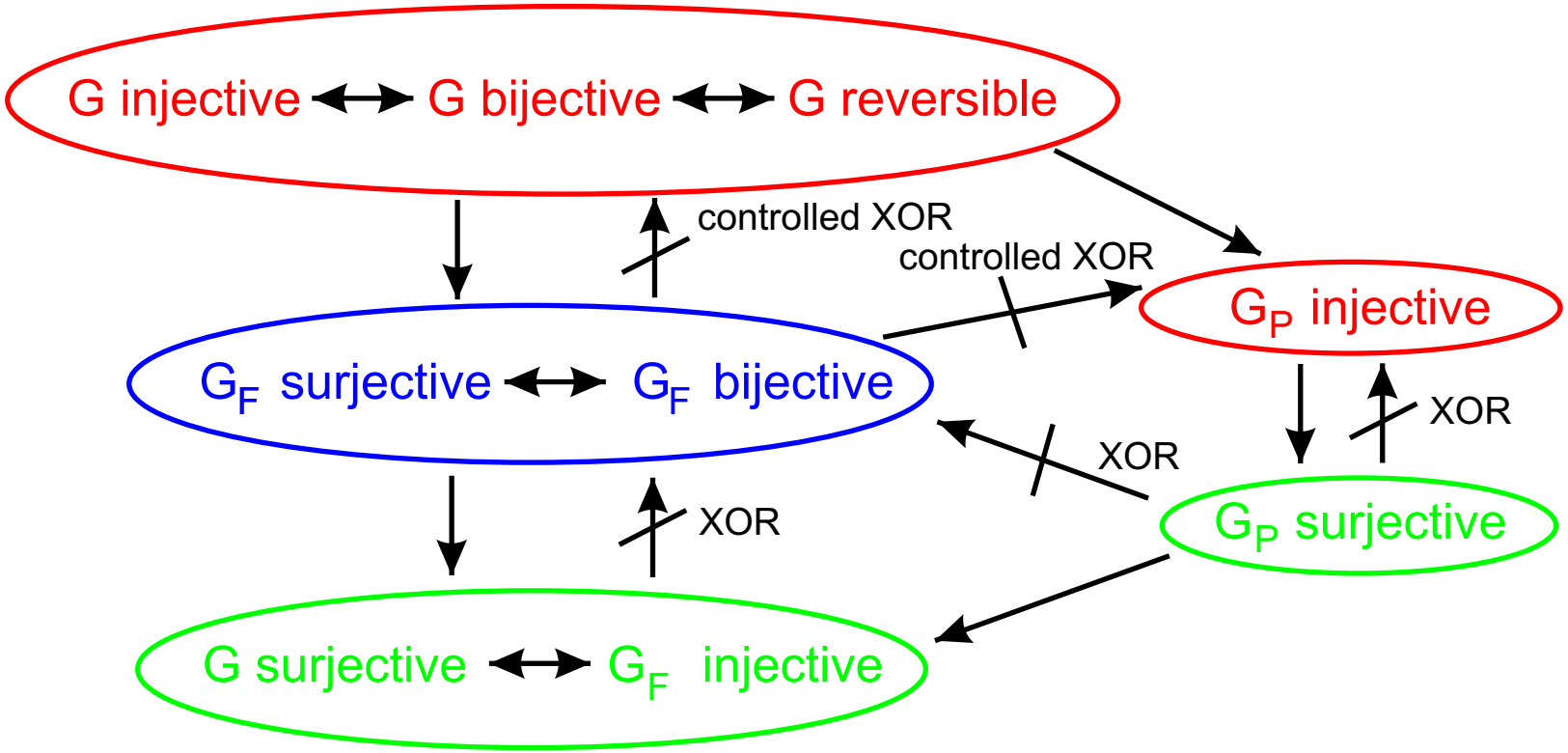

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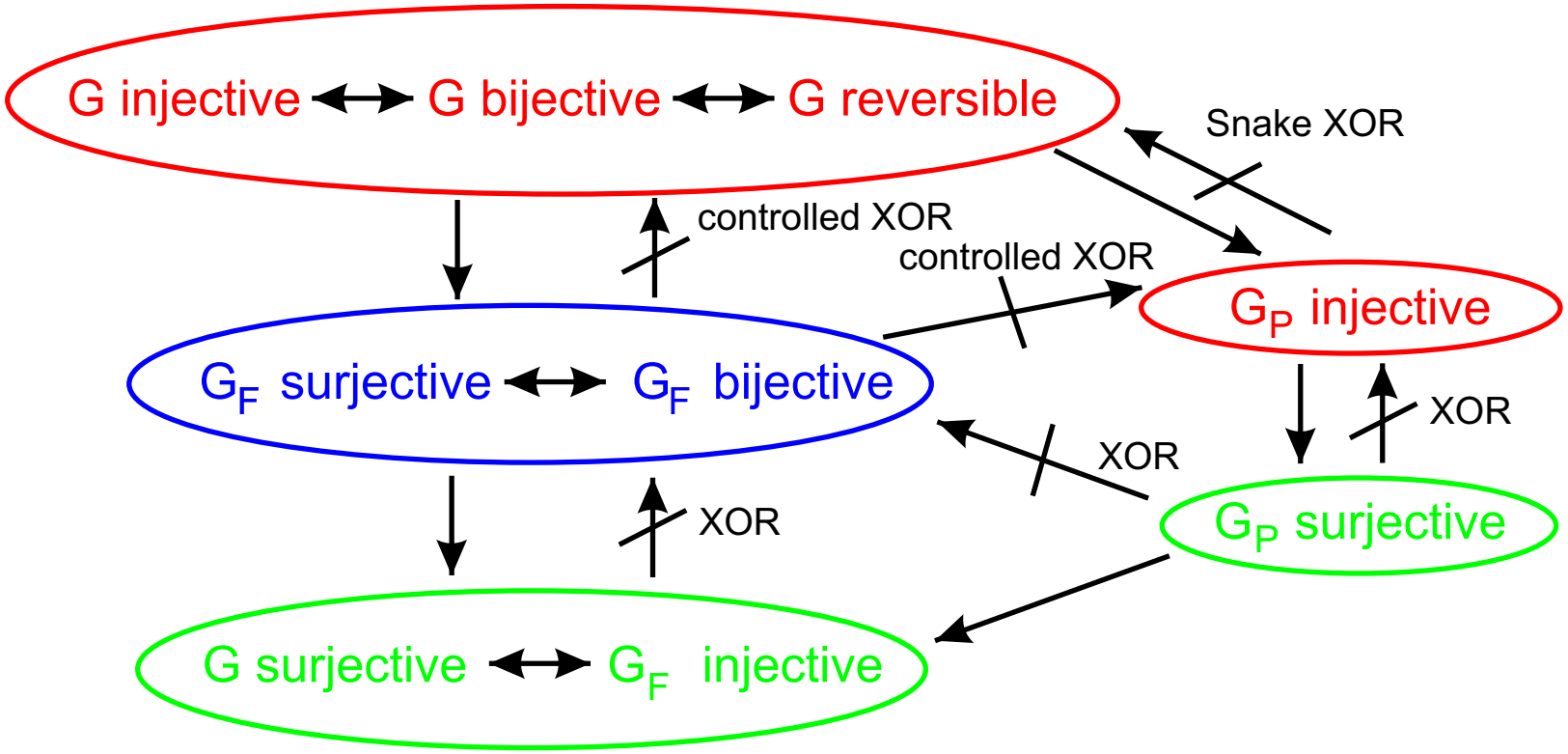
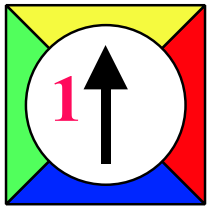


The **Snake XOR** CA is similar to the **Controlled XOR** CA except that the control layer is based on the tile set SNAKES.

The state set of the CA is

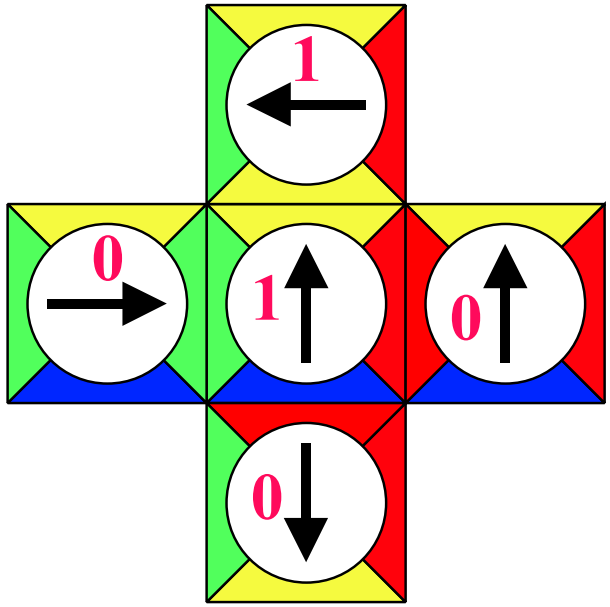
$$S = \text{SNAKES} \times \{0, 1\}.$$

(Each snake tile is attached a red bit.)



The local rule checks whether the tiling is valid at the cell:

- If there is a tiling error, no change in the state.



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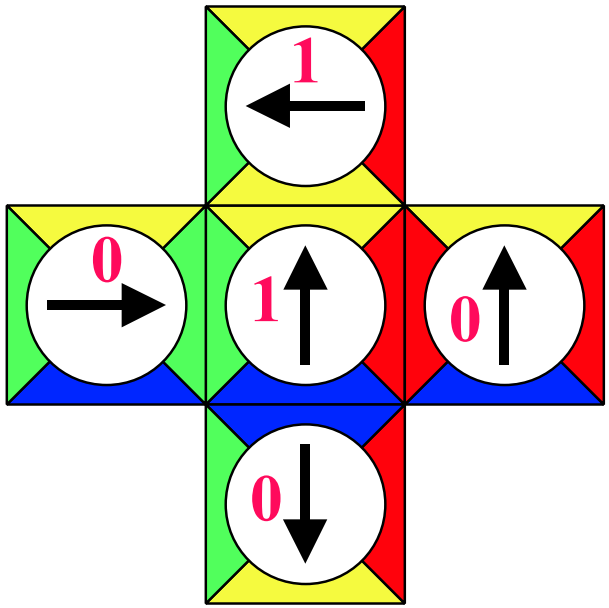

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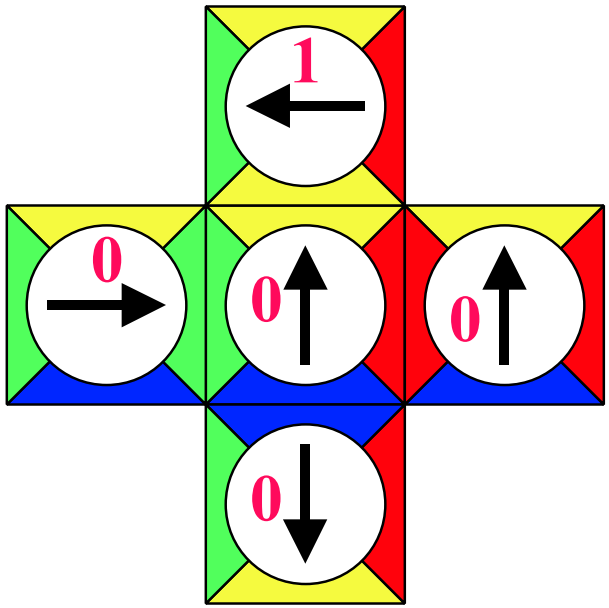

Snake XOR is not injective:

The following two configurations have the same successor: The SNAKES tilings of the configurations form the same valid tiling of the plane. In one of the configurations all bits are set to 0, and in the other configuration all bits are 1.

All cells are active because the tilings are correct. This means that all bits in both configurations become 0. So the two configurations become identical. The CA is not injective.

Snake XOR is injective on periodic configurations:

Suppose there are different periodic configurations c and d with the same successor. Since only bits may change, c and d must have identical SNAKES tiles everywhere. So they must have different bits 0 and 1 in some position $\vec{p}_1 \in \mathbb{Z}^2$.

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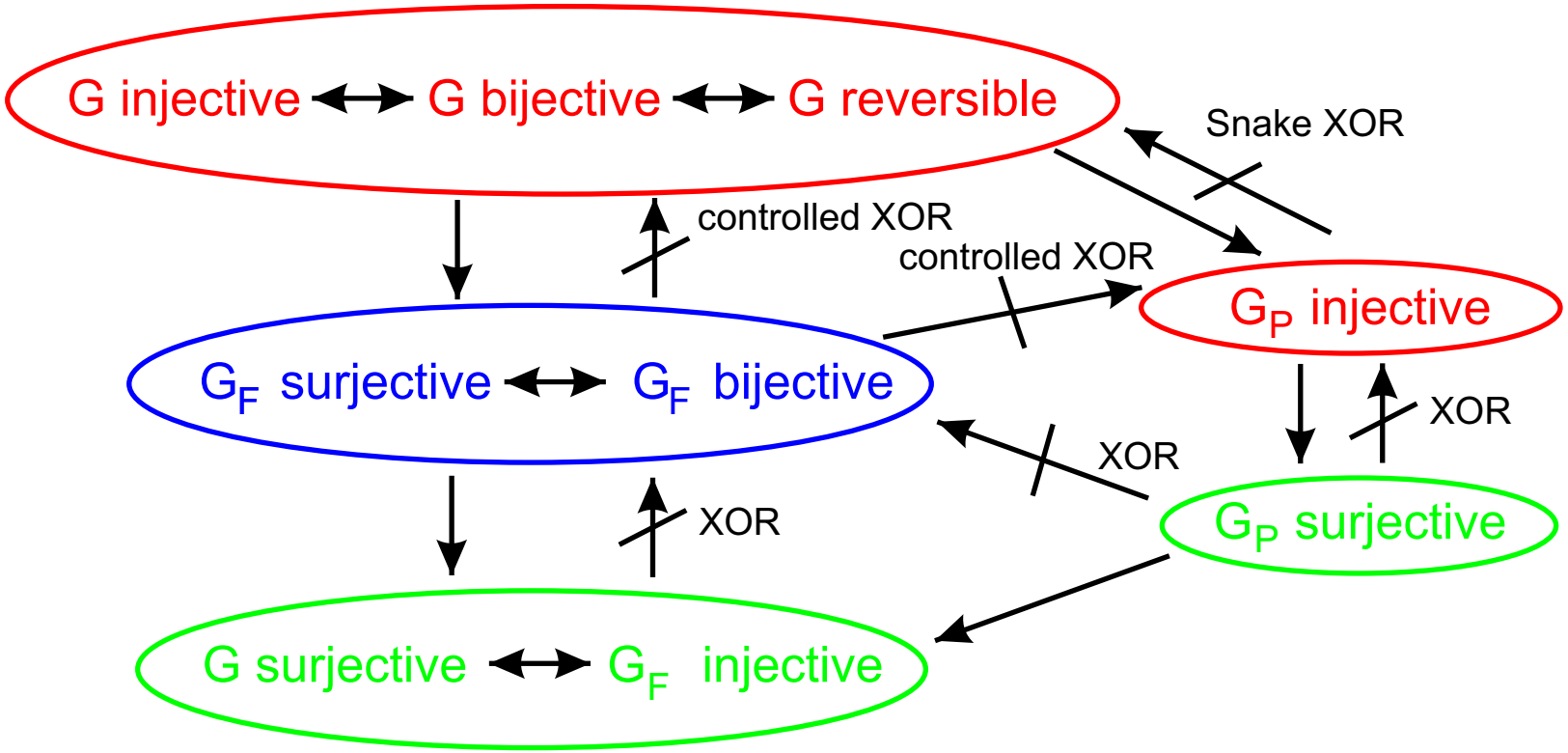
Because c and d have identical successors, the cell in position $\vec{p}_1$ must be active, that is, the SNAKES tiling is valid in position $\vec{p}_1$. Moreover, the bits stored in the neighboring position $\vec{p}_2$ (pointed by the arrow) are different in c and d. Hence we can repeat the reasoning in position $\vec{p}_2$.

The same reasoning can be repeated over and over again. The positions $\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots$ form a path that follows the arrows on the tiles. There is no tiling error at any tile on this path.

But this contradicts the fact that the plane filling property of SNAKES guarantees that on periodic tiling every path encounters a tiling error.


G injective ↔ G bijective ↔ G reversible
Snake XOR
controlled XOR
controlled XOR
G_P injective
G_F surjective ↔ G_F bijective
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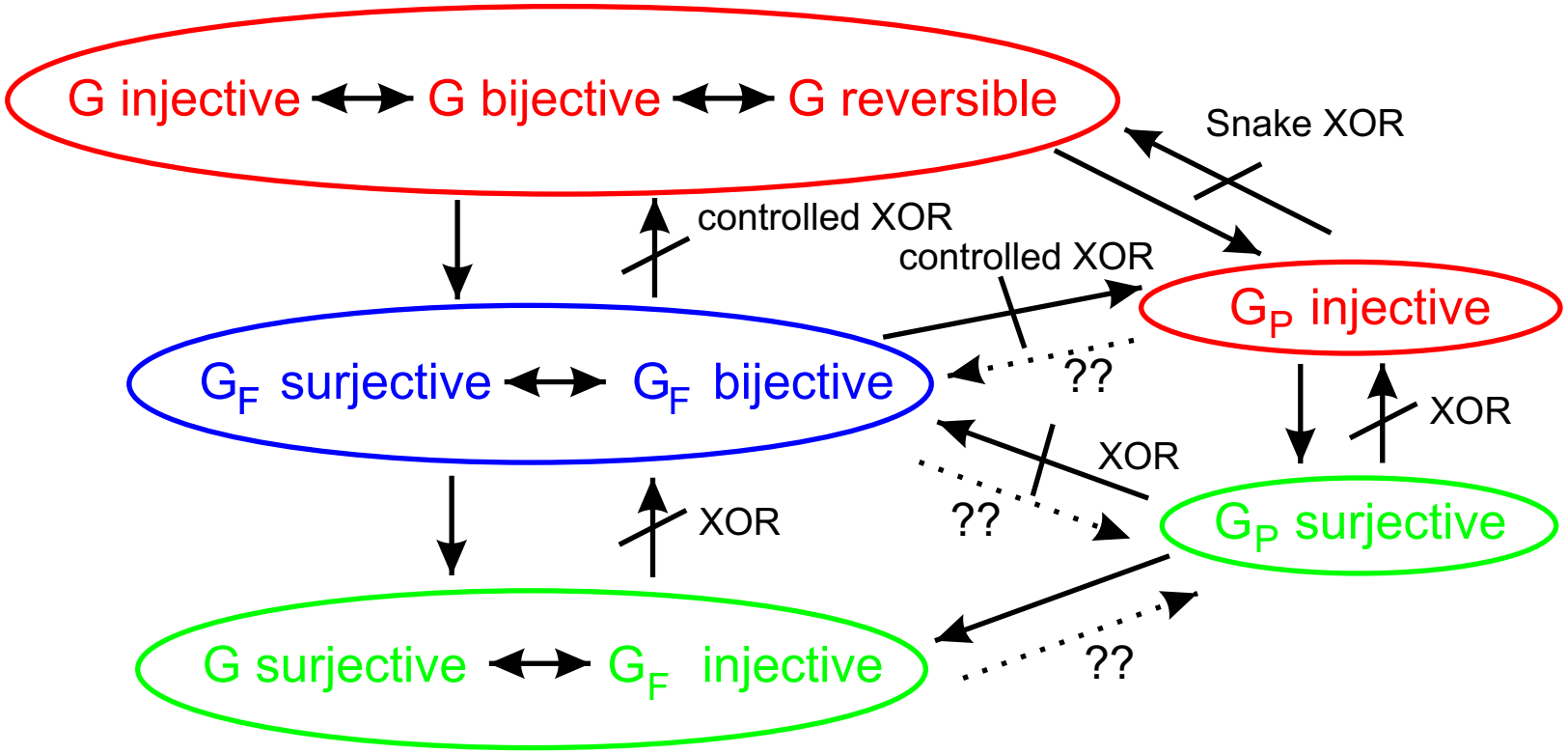

There are three implications whose status is unknown!

Second application of SNAKES: It is undecidable to determine if a given two-dimensional CA is reversible.

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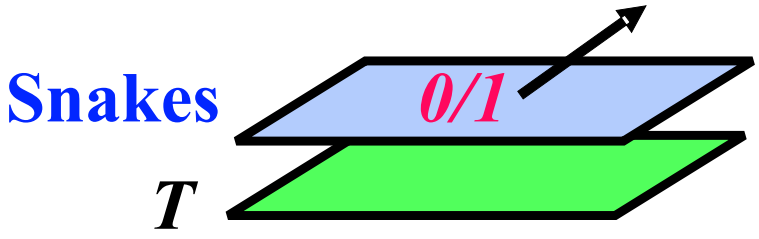
The proof is a reduction from the tiling problem, using the tile set SNAKES.

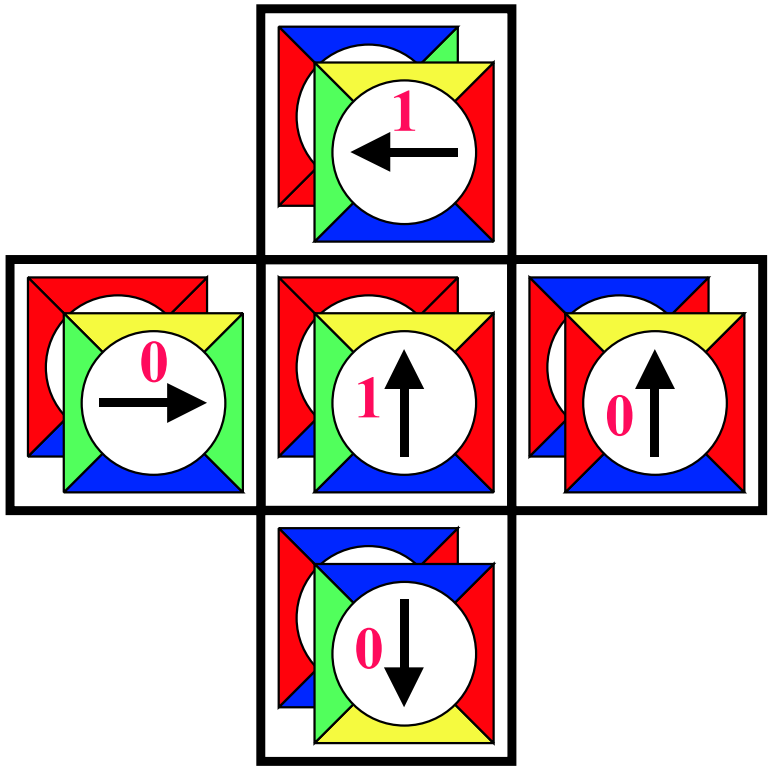
For any given tile set T we construct a CA with the state set

$$S = T \times \text{SNAKES} \times \{0, 1\}.$$



The local rule is analogous to **Snake XOR** with the difference that the correctness of the tiling is checked in both tile layers:

- If there is a tiling error then the cell is inactive.



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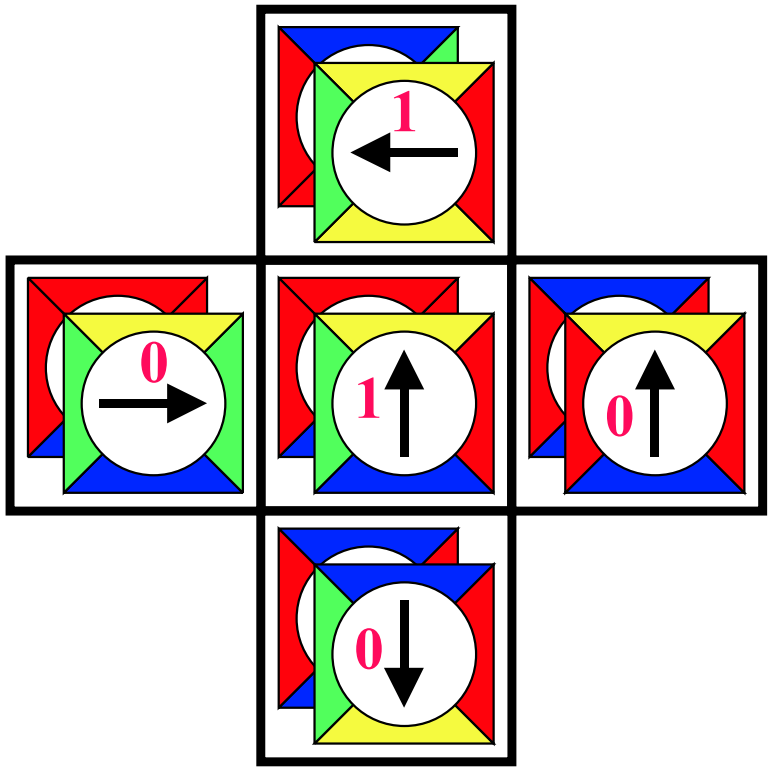

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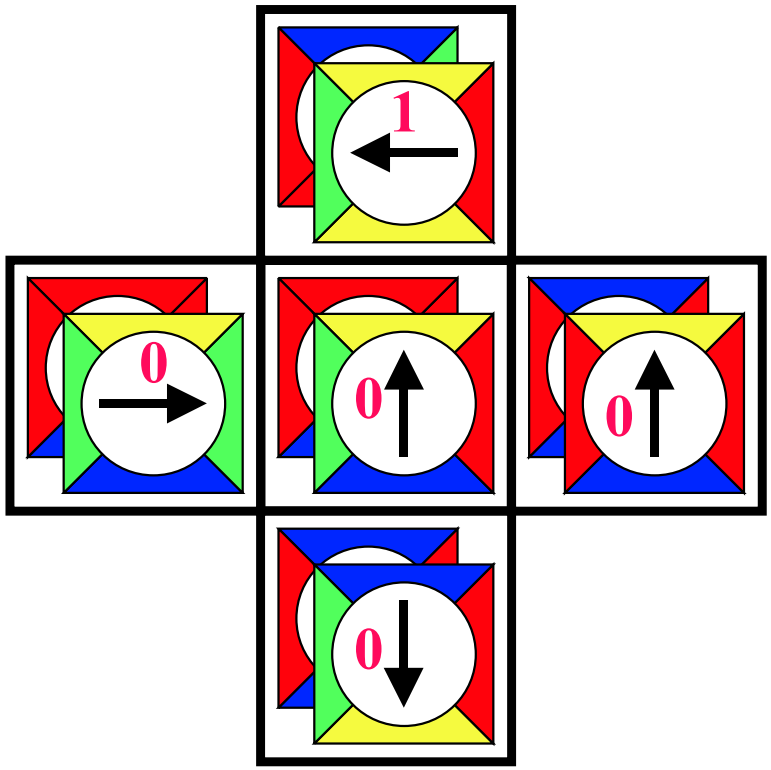
- If both tilings are correct at the cell then the bit of the cell is updated using the XOR rule with the neighboring cell in the direction of the arrow.



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We can reason exactly as with **Smake XOR**, and show that the CA is reversible if and only if the tile set T does not admit a plane tiling:

($\Longrightarrow$) If a valid tiling of the plane exists then we can construct two different configurations of the CA that have the same image under G. The SNAKES and the T layers of the configurations form the same valid tilings of the plane. In one of the configurations all bits are 0, and in the other configuration all bits are 1.

All cells are active because the tilings are correct. This means that all bits in both configurations become 0. So the two configurations become identical. The CA is not injective.

($\Longleftarrow$) Conversely, assume that the CA is not injective. Let c and d be two different configurations with the same successor. Since only bits may change, c and d must have identical SNAKES and T layers. So they must have different bits 0 and 1 in some position $\vec{p}_1 \in \mathbb{Z}^2$.

($\Longleftarrow$) Conversely, assume that the CA is not injective. Let c and d be two different configurations with the same successor. Since only bits may change, c and d must have identical SNAKES and T layers. So they must have different bits 0 and 1 in some position $\vec{p}_1 \in \mathbb{Z}^2$.

Because c and d have identical successors, the cell in position $\vec{p}_1$ must be active, that is, the tilings with both tile components are valid in position $\vec{p}_1$. Moreover, the bits stored in the neighboring position $\vec{p}_2$ (pointed by the arrow) are different in c and d. Hence we can repeat the reasoning in position $\vec{p}_2$.

The same reasoning can be repeated over and over again. The positions $\vec{p}_1, \vec{p}_2, \vec{p}_3, \ldots$ form a path that follows the arrows on the tiles. There is no tiling error at any tile on this path so the special property of SNAKES forces the path to cover arbitrarily large squares.

Hence T admits tilings of arbitrarily large squares, and consequently a tiling of the infinite plane. □

Theorem: It is undecidable whether a given two-dimensional CA is injective.

An analogous (but simpler!) construction can be made for the surjectivity problem, based on the fact surjectivity is equivalent to injectivity on finite configurations:

Theorem: It is undecidable whether a given two-dimensional CA is surjective.

Both problems are semi-decidable in one direction:

Injectivity is semi-decidable: Enumerate all CA G one-by one and check if G is the inverse of the given CA. Halt once (if ever) the inverse is found.

Non-surjectivity is semi-decidable: Enumerate all finite patterns one-by-one and halt once (if ever) an orphan is found.

From the undecidability of injectivity we obtain the interesting fact that there are some reversible CA that use von Neumann neighborhood but whose inverse automata use a very large neighborhood: There can be no computable upper bound on the extend of this inverse neighborhood.

So while topological arguments show that a finite neighborhood is enough to determine the previous state of a cell, computation theory shows that this neighborhood may be extremely large.

From the undecidability of surjectivity we see that that there are non-surjective CA whose smallest orphan is very large: There can be no computable upper bound on the extend of the smallest orphan.

So while the smallest known orphan for Game-Of-Life is pretty big (113 cells), this pales in comparison with some other CA.

Both reversibility and surjectivity can be easily decided among one-dimensional CA:

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We also know the tight bound on the extend of the inverse neighborhood: The neighborhood of a radius-$\frac{1}{2}$ injective CA with n states consists of at most $n-1$ consecutive cells.

We do not know any polynomial upper bound on the length of the smallest orphan for a radius-$\frac{1}{2}$ non-surjective CA with n states.

A CA G is called **periodic** if all configurations are temporally periodic. In this case there is a positive integer n such that G^n is the identity function.

The CA we constructed in the undecidability proof for reversibility is periodic when reversible. Hence we have

Theorem: It is undecidable whether a given two-dimensional CA is periodic.

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Theorem: It is undecidable whether a given two-dimensional CA is periodic.

Recently (in 2008) with N.Ollinger we showed that periodicity is also undecidable among one-dimensional CA:

Theorem: It is undecidable whether a given one-dimensional CA is periodic.

Outline of the talk(s)

(1) **Definitions** (configurations, neighborhoods, cellular automata, reversibility, injectivity, surjectivity)

(2) **Classical results** (Hedlund's theorem, balance in surjective CA, Garden-Of-Eden theorem)

(3) **Reversible CA** (universality, billiard ball CA)

(4) **Decidability questions** (Wang tiles, domino problem, snake tiles, undecidability of nilpotency, reversibility, surjectivity, periodicity)

(5) **Dynamical systems aspects** (equicontinuity classification, limit sets)

A topological dynamical system is a pair (X, F) where X is a compact space and $F : X \longrightarrow X$ is continuous. We know that $(S^{\mathbb{Z}^d}, G)$ is such a dynamical system, so concepts and methods of topological dynamics can be applied to cellular automata.

In the study of chaotic behavior one is interested in the propagation of errors during iterations of the system. This is also relevant to simulations by cellular automata: iterations of the CA inside a finite window may not give a reliable picture if the system is sensitive to changes in states outside the observation window.

The results reported here are proved for one-dimensional CA. Nevertheless, the concepts can be defined for any dimensionality d.

Configuration $c \in S^{\mathbb{Z}^d}$ is a **point of equicontinuity** if for every finite window $M \subseteq \mathbb{Z}^d$ there is another finite window $N \subseteq \mathbb{Z}^d$ such that changes in configuration c outside window N can have no effect inside M at any time:

$$x_{|N} = c_{|N} \Longrightarrow \forall t \geq 0 \ : \ G^t(x)_{|M} = G^t(c)_{|M}.$$

A CA is called **equicontinuous** if all configurations are points of equicontinuity.

Equicontinuous CA can be reliably simulated up to any precision.

Unfortunately, only trivial CA can be equicontinuos:

Theorem (Kurka): A CA is equicontinuous if and only if it is eventually periodic, that is, there is a preperiod m and a period $p > 0$ such that $G^{m+p} = G^m$.

All surjective CA that are eventually periodic are periodic, so we have:

Theorem (Blanchard and Tisseur): A surjective CA is equicontinuous if and only if it is periodic.

As periodicity is undecidable, we see that equicontinuity is an undecidable property, even among one-dimensional injective CA.

We call a CA **almost equicontinuous** if there are some points of equicontinuity.

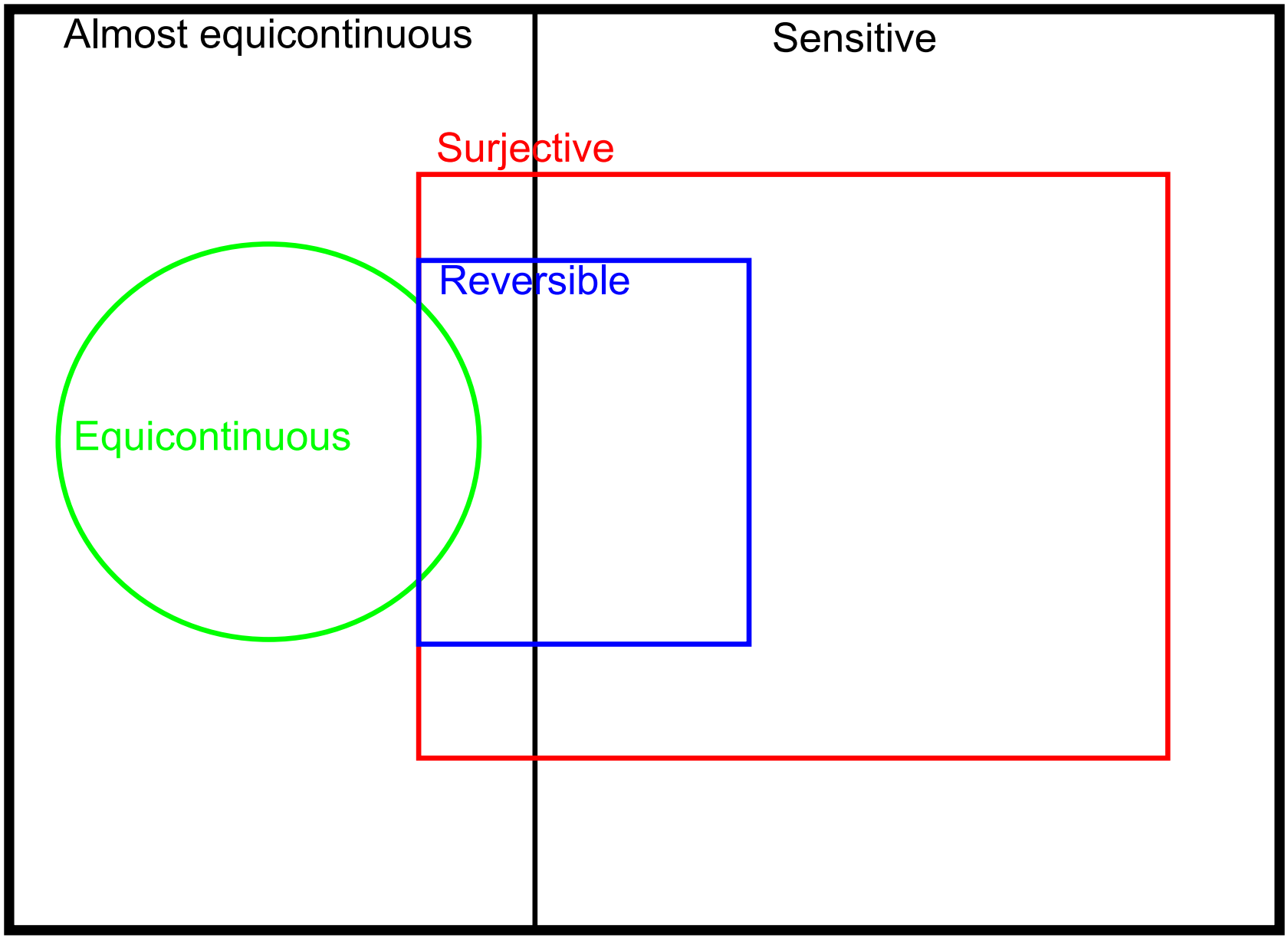
A CA is **sensitive (to initial conditions)** if there is a finite observation window $M \subseteq \mathbb{Z}^d$ such that differences to any configuration arbitrarily far away from M can propagate inside M.

More precisely, for all configurations c and all finite windows N there is a configuration e that agrees with c inside window N but $G^t(c)$ and $G^t(e)$ disagree inside window M, for some t.

It is clear that if the CA has points of equicontinuity then it cannot be sensitive, but also the converse is true:

Theorem A CA is sensitive if and only if it has no points of equicontinuity.

It was proved by B.Durand, et.al. that it is undecidable whether a given one-dimensional CA is sensitive.


Almost equicontinuous
Sensitive
Surjective
Reversible
Equicontinuous

A CA is called **positively expansive** if there exists a finite observation window $M \subseteq \mathbb{Z}^d$ such that all differences in configurations eventually propagate inside M. This is a very strong form of sensitivity.

More precisely, positive expansivity means that for all configurations $c \neq e$ there is time t such that $G^t(c)$ and $G^t(e)$ differ inside window M.

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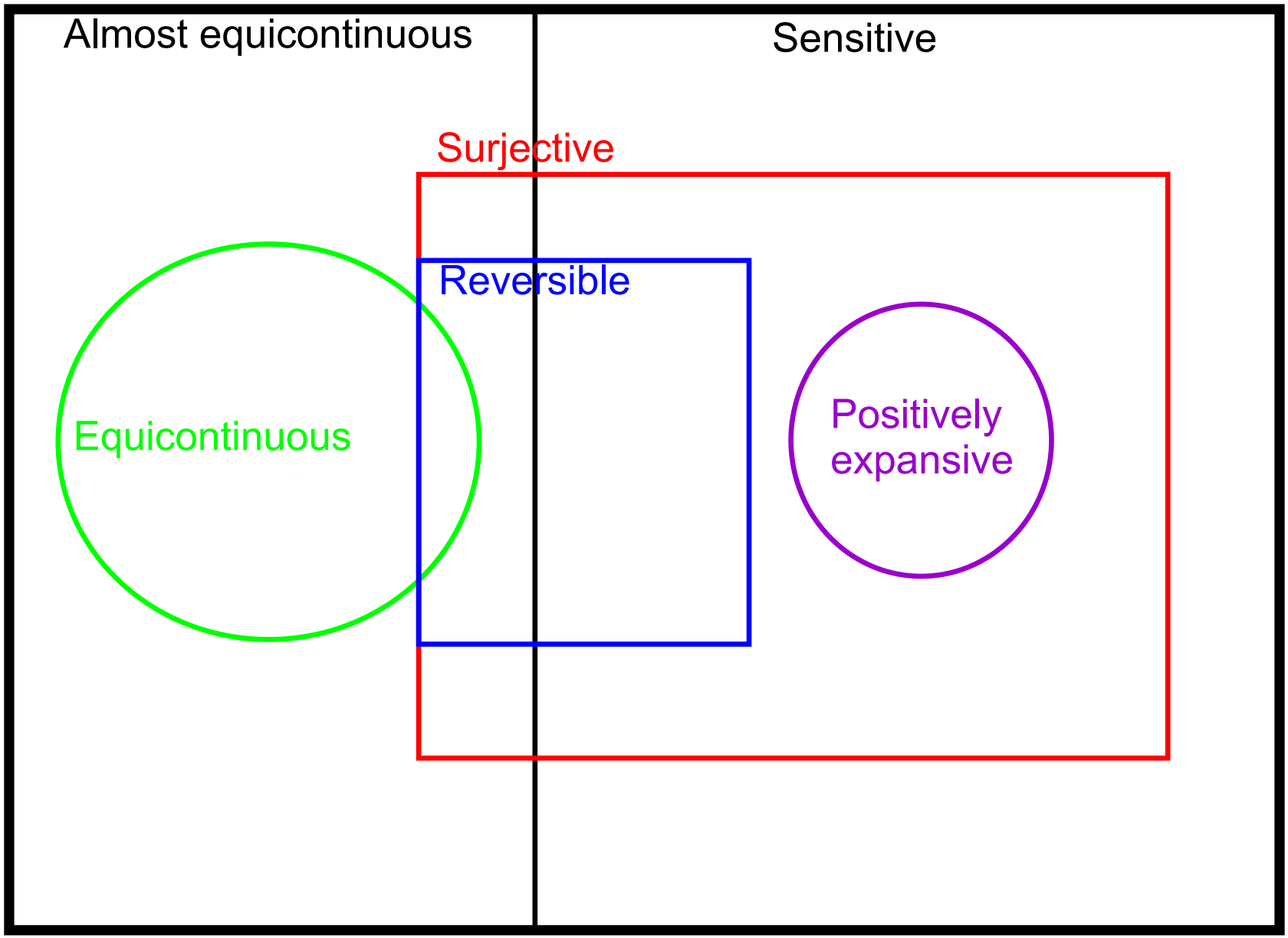
More precisely, positive expansivity means that for all configurations $c \neq e$ there is time t such that $G^t(c)$ and $G^t(e)$ differ inside window M.

There are no two- or higher dimensional positively expansive CA, there are no reversible positively expansive CA, while all positively expansive CA are surjective and sensitive to initial conditions.

Open problem: Is it decidable whether a given one-dimensional CA is positively expansive ?


Almost equicontinuous
Sensitive
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expansive

Based on these concepts Kurka proposed an equicontinuity classification of one-dimensional cellular automata into four classes, in the increasing order of sensitivity to changes in the initial configuration:

(K1) Equicontinuous CA,

(K2) Almost equicontinuous but non-equicontinuous CA,

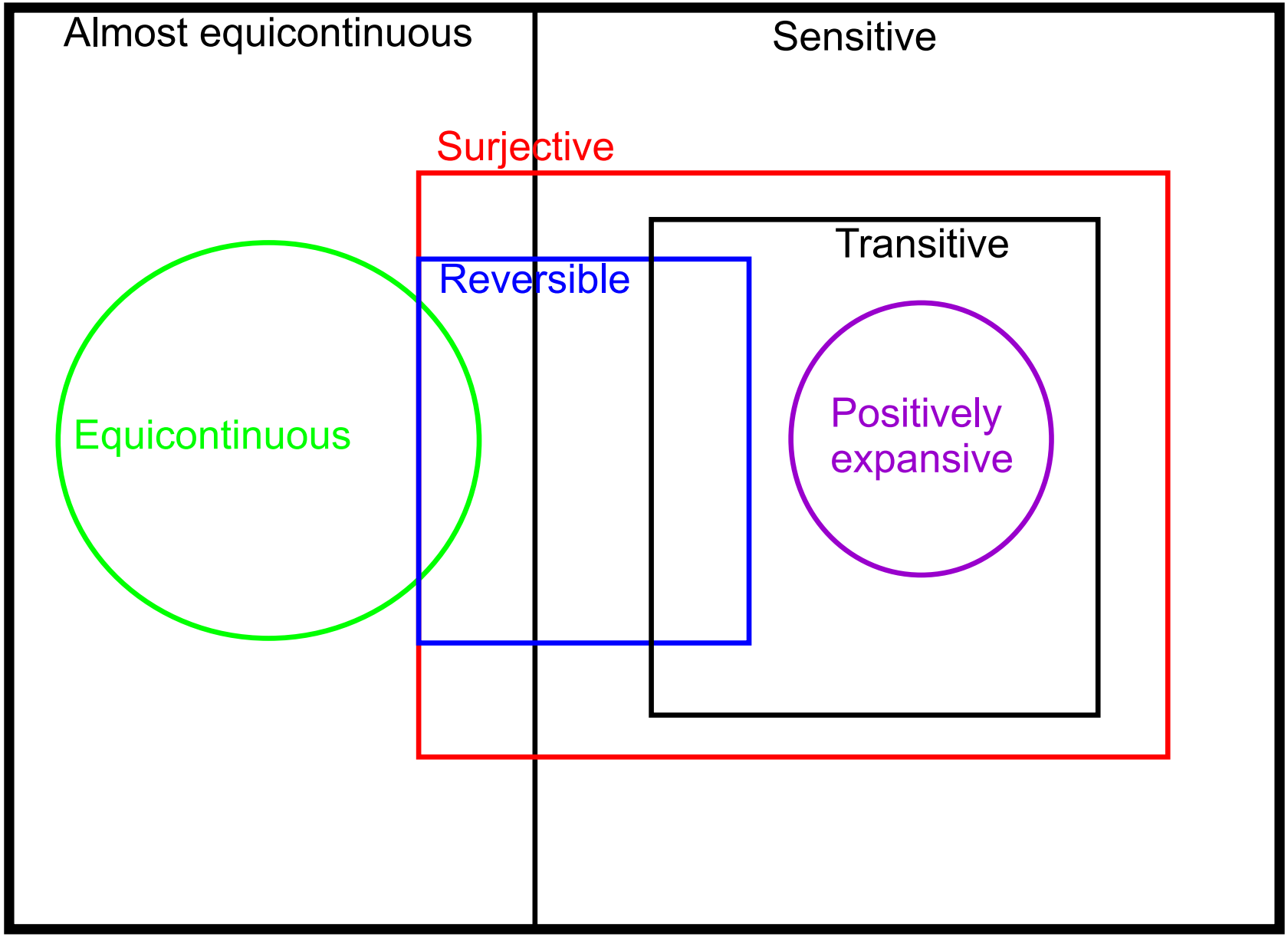
(K3) Sensitive but not positively expansive CA,

(K4) Positively expansive CA.

There are surjective CA in all four classes. There are reversible CA only in the first three classes.

Finally, a property dealing with mixing of configurations can be related to the equicontinuity classification: A CA is called **transitive** if for any two cylinders U and V there exists time t such that $G^t(U) \cap V \neq \emptyset$.

Theorem (Blanchard, Maass) and Kurka: Every positive expansive CA is transitive. Every transitive CA is surjective and sensitive.


Almost equicontinuous
Sensitive
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expansive

Different definitions of **chaotic systems** exist. A widely used definition by Devaney defines a dynamical system to be chaotic if

(1) it is transitive,

(2) temporally periodic points are dense,

(3) it is sensitive to initial conditions.

Because in CA transitivity implies sensitivity so only conditions (1) and (2) remain.

Different definitions of **chaotic systems** exist. A widely used definition by Devaney defines a dynamical system to be chaotic if

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Because in CA transitivity implies sensitivity so only conditions (1) and (2) remain.

Open problem: Are temporally periodic configurations dense when G is surjective ?

(If so then transitivity is equivalent to Devaney's chaos.)

Limit sets. . . under construction. . .

Conclusion

There are many topics not covered in this tutorial, many advanced results were not discussed, and many variants of the basic definitions were not mentioned. Topics left out include, for example,

- language recognition by cellular automata,
- linear cellular automata (=additive rules),
- intrinsic universality in cellular automata,
- error resilience in cellular automata
- asynchronous and stochastic CA,
- quantum CA,

- CA on Cayley graphs,
- dynamics under alternative topologies,
- firing squad synchronization problem and other cellular algorithms,
- measure theoretic approach.

Some open problems

- Universality of elementary rule 54.
- Do following implications hold on two-dimensional CA:

  G surjective $\Longrightarrow$ G_P surjective

  G_F surjective $\Longrightarrow$ G_P surjective

  G_P injective $\Longrightarrow$ G_F surjective
- Are three- and higher dimensional RCA compositions of block permutations and translations of a track ?
- Are temporally periodic configurations dense on surjective CA ?
- Is it decidable if a given CA is positively expansive ?